Spin Geometry and Supergravity

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Contents

Lecture 1

Clifford Algebras and Spinors

1.1 Clifford algebra and Spin group

Let us consider an n-dimensional vector space V defined on a field $\mathbb{K}$, with a quadratic form $Q : V \times V \to \mathbb{K}$.
The dual space is defined by $V^* = \{T \,|\, T : V \to \mathbb{K}\}$.
Given a basis $\{X_a\}$ for V, the dual basis $\{e^a\}$ for V^* is defined as $e^a(X_b) = \delta^a_b$.
A tensor N of type (p,q) is a multilinear mapping

$$N : \underbrace{V^* \times V^* \times ... \times V^*}_{p} \times \underbrace{V \times V \times ... \times V}_{q} \longrightarrow \mathbb{K}$$

The space of such tensors is denoted by $T^p_q(V)$ and with the tensor product $\otimes$ it is the tensor algebra. When q is zero, we say that N has degree p.
The space of totally antisymmetric tensors of degree p is denoted by $\Lambda_p(V)$, and its elements are called p-forms. The space of p-forms has dimension $\dim \Lambda_p(V) = \binom{n}{p}$.
The space of exterior forms is $\Lambda(V) = \bigoplus_{p=0}^{n} \Lambda_p(V)$, where $\Lambda_0(V) = \mathbb{K}$ and $\Lambda_1(V) \cong V$.

We can define two different algebra structures on $\Lambda(V)$;

i) The exterior (wedge) product

$$\begin{aligned} \wedge : \Lambda_p(V) \times \Lambda_q(V) &\longrightarrow \Lambda_{p+q}(V) \\ \omega\,,\,\phi &\longmapsto \omega \wedge \phi = \frac{1}{2}\left(\omega \otimes \phi - \phi \otimes \omega\right) \end{aligned}$$

gives $\Lambda(V)$ the exterior algebra structure.

ii) The Clifford product

$$\begin{aligned} . : \Lambda_1(V) \times \Lambda_1(V) &\longrightarrow \Lambda(V) \\ x\,,\,y &\longmapsto x.y + y.x = 2Q(x,y) \end{aligned}$$

gives $\Lambda(V)$ the Clifford algebra structure which is denoted by $Cl(V,Q)$.

We can define two automorphisms on $\Lambda(V)$;

i) The automorphism $\eta : \Lambda_p(V) \longrightarrow \Lambda_p(V)$ with $\eta^2 = 1$ is defined by $\eta\omega = (-1)^p\omega$ and we have $\eta(\omega \circ \phi) = \eta\omega \circ \eta\phi$ for $\omega, \phi \in \Lambda_p(V)$ and $\circ$ denotes the wedge or Clifford product.

ii) The anti-automorphism $\xi : \Lambda_p(V) \longrightarrow \Lambda_p(V)$ with $\xi^2 = 1$ is defined by $\xi\omega = (-1)^{\lfloor p/2 \rfloor}\omega$ and we have $\xi(\omega \circ \phi) = \xi\phi \circ \xi\omega$ for $\omega, \phi \in \Lambda_p(V)$, $\circ$ denotes the wedge or Clifford product and $\lfloor\rfloor$ is the floor function which takes the integer part of the argument.

By the automorphism η, the Clifford algebra gains a $\mathbb{Z}_2$-grading

$$Cl(V,Q) = Cl^0(V,Q) \oplus Cl^1(V,Q)$$

where for $\phi \in Cl^0(V,Q)$ we have $\eta\phi = \phi$ and for $\phi \in Cl^1(V,Q)$ we have $\eta\phi = -\phi$. So, $Cl(V,Q)$ is a $\mathbb{Z}_2$-graded algebra or a superalgebra with the even part $Cl^0(V,Q)$ and the odd part $Cl^1(V,Q)$.

Certain elements of $Cl(V,Q)$ are invertible. For example, from the definition of the Clifford product, one has $x^2 = Q(x,x)$ for $x \in V$.
If $Q(x,x) \neq 0$, then we have $x^{-1} = \frac{x}{Q(x,x)}$.

The group of invertible elements in $Cl(V,Q)$ is denoted by

$$Cl^\times(V,Q) = \left\{\phi \in Cl(V,Q) \,|\, \phi\phi^{-1} = \phi^{-1}\phi = 1\right\}.$$

The Clifford group is defined by

$$P(V,Q) = \left\{\phi \in Cl^\times(V,Q) \,|\, \phi.V.\phi^{-1} = V\right\}.$$

Pin group

$$Pin(V,Q) = \left\{\phi \in P(V,Q) \,|\, \phi^\xi\phi = 1\right\}.$$

Spin group

$$Spin(V,Q) = Pin(V,Q) \cap Cl^0(V,Q).$$

Pin group is the double cover of the orthogonal group (rotations)

$$Pin(V,Q) \xrightarrow{2:1} O(V,Q)$$

Spin group is the double cover of the special orthogonal group (rotations without reflections)

$$Spin(V,Q) \xrightarrow{2:1} SO(V,Q)$$

Representations of the Pin group are called pinors, $\rho : Pin(V,Q) \to \mathrm{End}\,\Xi$ where Ξ is the pinor space.
Representations of the Spin group are called spinors, $\rho : Spin(V,Q) \to \mathrm{End}\,\Sigma$ where Σ is the spinor space.

1.2 Clifford and spinor bundles

Let us consider an n-manifold M with metric g, tangent bundle TM and the cotangent bundle T^*M.
We denote the frame basis on TM as $\{X_a\}$ and co-frame basis on T^*M as e^a with the property $e^a(X_b) = \delta^a_b$.
We can define the p-form bundle $\Lambda^p M$ on M with the wedge product $\wedge : \Lambda^p M \times \Lambda^q M \longrightarrow \Lambda^{p+q} M$. We also have the exterior bundle ΛM on M.
From the Levi-Civita connection ∇ on M, we can define related differential operators on ΛM;
Exterior derivative $d : \Lambda^p M \longrightarrow \Lambda^{p+1} M$ with $d = e^a \wedge \nabla_{X_a}$
Co-derivative $\delta : \Lambda^p M \longrightarrow \Lambda^{p-1} M$ with $\delta = -i_{X^a} \nabla_{X_a}$
where $i_X : \Lambda^p M \longrightarrow \Lambda^{p-1} M$ is the interior derivative (contraction) operation defined by

$$\omega(X_1, ..., X_p) = i_{X_1}\omega(X_2, ..., X_p)$$

for $\omega \in \Lambda^p M$. We can also define the Hodge star operation $* : \Lambda^p M \longrightarrow \Lambda^{n-p} M$ as

$$*(e^{a_1} \wedge ... \wedge e^{a_p}) = \frac{1}{(n-p)!}\epsilon^{a_1 \dots a_p}{}_{a_{p+1} \dots a_n} e^{a_{p+1}} \wedge ... \wedge e^{a_n}.$$

If we change the product rule for e^a from the wedge product to the Clifford product

$$e^a.e^b + e^b.e^a = 2g^{ab}$$

then we have the Clifford bundle $Cl(M)$.

There is a relation between Clifford and exterior bundles;
The action of $Cl(M)$ on ΛM is given for $x \in \Lambda^1 M$ and $\omega \in \Lambda^p M$ as

$$\begin{aligned} x.\omega &= x \wedge \omega + i_{\widetilde{x}}\omega \\ \omega.x &= x \wedge \eta\omega - i_{\widetilde{x}}\eta\omega \end{aligned}$$

where $\widetilde{x}$ is the vector field metric dual to x which is defined by $x(Y) = g(\widetilde{x}, Y)$ for $Y \in TM$.

Although the Spin group $Spin(M)$ is a subgroup in $Cl(M)$, there are topological obstructions to define a spin bundle on M.
In general, a basis $\{X_a\}$ of TM transforms to another basis under the group $GL(n)$ (this is called as $GL(n)$-structure).
If $\{X_a\}$ is an orthogonal basis, then it transforms under the rotation group $O(n)$ ($O(n)$-structure).
For a bundle $\pi : E \to M$ on M, one can define some characteristic classes to characterize its topological structure. A bundle E is always locally trivial $\pi : E = F \times M \to M$, but globally it can be non-trivial. The non-triviality of a bundle is measured by the characteristic classes.
For the tangent bundle $TM \to M$;
If the first Stiefel-Whitney class vanishes $w_1(M) = 0$, then one can define an orientation on M and the frame basis $\{X_a\}$ transforms under the group $SO(n)$ ($SO(n)$-structure).

If the second Stiefel-Whitney class also vanishes $w_2(M) = 0$, then one can define a lifting of $SO(n)$-structure to $Spin(n)$-structure by $Spin(n) \xrightarrow{2:1} SO(n)$.
Then, it is said that the manifold M has a **spin structure** and M is called as a **spin manifold**.

So, in that case, one can define a spinor bundle ΣM on M induced from the Clifford bundle $Cl(M)$.
Let Σ denotes the representation space of $Spin(n)$,
the associated bundle of the $Spin(n)$ bundle $P \to M$ which is defined by

$$\Sigma M = P \times_{Spin(n)} \Sigma$$

is called the spinor bundle $\Sigma M \to M$ on M.
The sections of the spinor bundle are called spinor fields.

One can also define an action of $Cl(M)$ on ΣM
For $\phi \in Cl(M)$ and $\psi \in \Sigma M$

$$\begin{array}{rcl} c : CL(M) & \longrightarrow & \Sigma M \\ \phi\,, & \longmapsto & c(\phi, \psi) = \phi.\psi \end{array}$$

by the Clifford multiplication.

Lecture 2

Twistor and Killing Spinors

2.1 Dirac and twistor operators

Let us consider the Levi-Civita connection ∇ on TM.
∇ can be induced onto the bundles ΛM, $Cl(M)$ and ΣM.
So, we have the connection on spinor fields induced from the Levi-Civita connection;

$$\begin{array}{rcl} \nabla : \Sigma M & \longrightarrow & T^*M \otimes \Sigma M \\ \psi & \longmapsto & e^a \otimes \nabla_{X_a}\psi \end{array}$$

From the Clifford action on ΣM, for $T^*M \cong \Lambda_1 M \subset Cl(M)$ we have

$$\begin{array}{rcl} c : T^*M \otimes \Sigma M & \longrightarrow & \Sigma M \\ \widetilde{X} \otimes \psi & \longmapsto & \widetilde{X}.\psi \end{array}$$

where $\widetilde{X}$ is the 1-form metric dual of the vector field X. Note that the composition of the Clifford action and the connection

$$\begin{array}{rcl} c \circ \nabla : \Sigma M & \xrightarrow{\nabla} & T^*M \otimes \Sigma M \xrightarrow{c} \Sigma M \\ \psi & \mapsto & e^a \otimes \nabla_{X_a}\psi \mapsto e^a.\nabla_{X_a}\psi = \not{D}\psi \end{array}$$

corresponds to the Dirac operator. However, this is not the only first-order differential operator that we can define on spinor fields.

We can write $T^*M \otimes \Sigma M$ as a direct sum of two components

$$T^*M \otimes \Sigma M = S_1 \oplus S_2$$

by defining two projection operators P_1 and P_2 that satisfy $P_1 + P_2 = \mathbb{I}$;

$$\begin{array}{rcl} P_1 : T^*M \otimes \Sigma M & \longrightarrow & S_1 \\ \widetilde{X} \otimes \psi & \longmapsto & \frac{1}{n} e^a \otimes (e_a.\widetilde{X}.\psi) \end{array}$$

and

$$\begin{array}{rcl} P_2 : T^*M \otimes \Sigma M & \longrightarrow & S_2 \\ \widetilde{X} \otimes \psi & \longmapsto & \widetilde{X} \otimes \psi - \dfrac{1}{n} e^a \otimes (e_a.\widetilde{X}.\psi) \end{array}$$

Let us take the compositions of P_1 and P_2 with c

$$\begin{array}{rcl} c \circ P_1 : T^*M \otimes \Sigma M & \xrightarrow{P_1} & S_1 \xrightarrow{c} \Sigma M \\ \widetilde{X} \otimes \psi & \mapsto & \dfrac{1}{n} e^a \otimes (e_a.\widetilde{X}.\psi) \mapsto \dfrac{1}{n} e^a.e_a.\widetilde{X}.\psi = \widetilde{X}.\psi \end{array}$$

and

$$\begin{array}{rcl} c \circ P_2 : T^*M \otimes \Sigma M & \xrightarrow{P_2} & S_2 \xrightarrow{c} \Sigma M \\ \widetilde{X} \otimes \psi & \mapsto & \widetilde{X} \otimes \psi - \dfrac{1}{n} e^a \otimes (e_a.\widetilde{X}.\psi) \mapsto \\ & \mapsto & \widetilde{X}.\psi - \dfrac{1}{n} e^a.e_a.\widetilde{X}.\psi = 0 \end{array}$$

where we have used $e^a.e_a = e^a \wedge e_a - i_{X^a} e_a = n$ since $e^a \wedge e_a = 0$ and $i_{X^a} e_a = e_a(X^a) = \delta^a_a = n$.
So, this implies that we have $S_1 = im(c)$ and $S_2 = ker(c)$.
Then, we have $T^*M \otimes \Sigma M = \Sigma M \oplus ker(c)$. We will call $S_1 = \Sigma M$ as spinor bundle and $S_2 = ker(c)$ as twistor bundle.

Now, we consider two projections of ∇ and see that we can define two first-order differential operators on ΣM.

i) The projection to the first component

$$\begin{array}{rcl} c \circ P_1 \circ \nabla : \Sigma M & \xrightarrow{\nabla} & T^*M \otimes \Sigma M \xrightarrow{P_1} S_1 \xrightarrow{c} \Sigma M \\ \psi & \mapsto & e^a \otimes \nabla_{X_a} \psi \mapsto \\ & \mapsto & \dfrac{1}{n} e^b \otimes (e_b.e^a.\nabla_{X_a} \psi) \mapsto \dfrac{1}{n} e^b.e_b.e^a.\nabla_{X_a} \psi = \not{D}\psi \end{array}$$

is the **Dirac operator**.
The spinors which are in the kernel of the Dirac operator

$$\not{D}\psi = 0$$

are called **harmonic spinors** and are solutions of the massless Dirac equation.
The spinors which ar eeigenspinors of the Dirac operator

$$\not{D}\psi = m\psi$$

are solutions of the massive Dirac equation, where m is a constant corresponding to mass.

ii) The projection to the second component

$$\begin{array}{rcl} c \circ P_2 \circ \nabla : \Sigma M & \overset{\nabla}{\longrightarrow} & T^*M \otimes \Sigma M \overset{P_2}{\longrightarrow} S_2 \overset{c}{\to} ker(c) \\ \psi & \mapsto & e^a \otimes \nabla_{X_a}\psi \mapsto \\ & \mapsto & e^a \otimes \nabla_{X_a}\psi - \dfrac{1}{n} e^b \otimes (e_b.e^a.\nabla_{X_a}\psi) \mapsto \\ & \mapsto & e^{\cdot}\nabla_{X_a}\psi - \dfrac{1}{n} e^b.e_b.e^a.\nabla_{X_a}\psi = 0 \end{array}$$

So, the projection onto S_2 gives $e^a \otimes \left(\nabla_{X_a}\psi - \frac{1}{n} e_a.\not{D}\psi\right)$. Then, we have the **twistor (Penrose) operator**

$$P_X := \nabla_X - \frac{1}{n}\widetilde{X}.\not{D}$$

for $X \in TM$ and $\widetilde{X} \in T^*M$ its metric dual.
The spinors ψ which are in the kernel of the twistor operator and satisfy the following equation

$$\nabla_X \psi = \frac{1}{n}\widetilde{X}.\not{D}\psi$$

are called **twistor spinors**.
If a spinor ψ is a solution of both massive Dirac equation and twistor equation, then it is called as a **Killing spinor** and satisfies the following equation

$$\nabla_X \psi = \lambda \widetilde{X}.\psi$$

where λ is the Killing number, wh,ch can be real or pure imaginary. From $\nabla_X\psi = \frac{1}{n}\widetilde{X}.\not{D}\psi = \dfrac{m}{n}\widetilde{X}.\psi$, we have $\lambda := \frac{m}{n}$.
The spinors which are in the kernel of ∇

$$\nabla_X \psi = 0$$

are called **parallel spinors** (or covariantly constant spinors). This is a special case $\lambda = 0$ of Killing spinors.

2.2 Integrability conditions

We investigate the integrability conditions for the existence of special types of spinors on a spin manifold M.
i) Schrödinger-Lichnerowicz (Weitzenböck) formula
From the definition of the Dirac operator, we have for any spinor ψ

$$\begin{array}{rcl} \not{D}^2\psi & = & e^b.\nabla_{X_b}\left(e^a.\nabla_{X_a}\psi\right) \\ & = & e^b.\left(\nabla_{X_b}e^a.\nabla_{X_a}\psi + e^a.\nabla_{X_b}\nabla_{X_a}\psi\right). \end{array}$$

We will use the normal coordinates for which connection coefficients are zero and we have $\nabla_{X_a} e_b = 0 = [X_a, X_b]$. So, we can write

$$\begin{aligned}
\not{D}^2\psi &= e^b.e^a.\nabla_{X_b}\nabla_{X_a}\psi \\
&= \frac{1}{2}(e^b.e^a + e^a.e^b).\nabla_{X_b}\nabla_{X_a}\psi + \frac{1}{2}(e^b.e^a - e^a.e^b).\nabla_{X_b}\nabla_{X_a}\psi \\
&= \nabla_{X^a}\nabla_{X_a}\psi + \frac{1}{4}(e^b.e^a - e^a.e^b).\left(\nabla_{X_b}\nabla_{X_a} - \nabla_{X_a}\nabla_{X_b}\right)\psi \\
&= \nabla_{X^a}\nabla_{X_a}\psi - \frac{1}{2}e^b.e^a.R(X_a, X_b)\psi
\end{aligned}$$

where we have divided the symmetric and antisymmetric parts and used the identities $e^b.e^a + e^a.e^b = 2g^{ab}$ and $e^a.e^b = -e^b.e^a$ for $a \neq b$ in the second and third lines. The curvature operator is defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ for $X, Y \in TM$. In normal coordinates, we have $R(X_a, X_b) = [\nabla_{X_a}, \nabla_{X_b}]$.
For any spinor ψ, the action of the curvature operator can be written in terms of the curvature 2-forms R_{ab} as

$$R(X_a, X_b)\psi = \frac{1}{2}R_{ab}.\psi$$

(The proof can be found in; I.M. Benn and R.W. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics*, 1987, Ch.8-9). So, we have

$$\not{D}^2\psi = \nabla_{X^a}\nabla_{X_a}\psi - \frac{1}{4}e^b.e^a.R_{ab}.\psi$$

We can define $\nabla^2 = \nabla_{X^a}\nabla_{X_a}$, Ricci 1-forms $P_a = i_{X^a}R_{ab}$ and the curvature scalar $\mathcal{R} = i_{X^a}P_a$.
For zero torsion, we have the identities $R_{ab} \wedge e^b = 0$ and $P_a \wedge e^a = 0$.
So, we can write the identities $e^a.R_{ab} = e^a \wedge R_{ab} + i_{X^a}R_{ab} = P_b$ and $e^b.P_b = e^b \wedge P_b + i_{X^b}P_b = \mathcal{R}$. Then, we have

$$\not{D}^2\psi = \nabla^2\psi - \frac{1}{4}\mathcal{R}\psi.$$

This is called the Schrödinger-Lichnerowicz (Weitzenböck) formula.

ii) Twistor spinors

The existence of special types of spinors constrain the geometry of M.
Let us consider a twistor spinor ψ, namely we have $\nabla_{X_a}\psi = \frac{1}{n}e_a.\not{D}\psi$.
By taking the second covariant derivative

$$\nabla_{X_b}\nabla_{X_a}\psi = \frac{1}{n}\nabla_{X_b}e_a.\not{D}\psi + \frac{1}{n}e_a.\nabla_{X_b}\not{D}\psi$$

in normal coordinates

$$\nabla_{X_b}\nabla_{X_a}\psi = \frac{1}{n}e_a.\nabla_{X_b}\not{D}\psi$$

and similarly for the reversed order of indices

$$\nabla_{X_a}\nabla_{X_b}\psi = \frac{1}{n}e_b.\nabla_{X_a}\not{D}\psi.$$

From the difference of the last two equations and the definition of the curvature operator, we obtain

$$R(X_a, X_b)\psi = \frac{1}{n}\left(e_b.\nabla_{X_a}\not{D}\psi - e_a.\nabla_{X_b}\not{D}\psi\right)$$

and the action of the curvature operator on spinors gives

$$R_{ab}.\psi = \frac{2}{n}\left(e_b.\nabla_{X_a}\not{D}\psi - e_a.\nabla_{X_b}\not{D}\psi\right). \tag{2.1}$$

By Clifford multiplying with e^a from the left (from $e^a.R_{ab} = P_b$)

$$P_b.\psi = \frac{2}{n}\left(e^a.e_b.\nabla_{X_a}\not{D}\psi - e^a.e_a.\nabla_{X_b}\not{D}\psi\right)$$

and using the identities $e^a.e_b + e_b.e^a = 2g^a_b$ and $e^a.e_a = n$

$$P_b.\psi = \frac{2}{n}\left(-e_b.e^a.\nabla_{X_a}\not{D}\psi + 2\nabla_{X_b}\not{D}\psi - n\nabla_{X_b}\not{D}\psi\right)$$

From the definition of the Dirac operator, we have

$$P_b.\psi = -\frac{2}{n}e_b.\not{D}^2\psi - \frac{2(n-2)}{n}\nabla_{X_b}\not{D}\psi. \tag{2.2}$$

By Clifford multiplying with e^b from the left (from $e^b.P_b = \mathcal{R}$)

$$\begin{aligned}\mathcal{R} &= -\frac{2}{n}e^b.e_b.\not{D}^2\psi - \frac{2(n-2)}{n}e^b.\nabla_{X_b}\not{D}\psi \\ &= -\frac{4(n-1)}{n}\not{D}^2\psi \end{aligned} \tag{2.3}$$

where we have used $e^b.e_b = n$ and the definition of the Dirac operator. Hence, we obtain the first integrability condition for twistor spinors

$$\not{D}^2\psi = -\frac{n}{4(n-1)}\mathcal{R}\psi \tag{2.4}$$

By substituting this equality in (2.2), we have

$$\nabla_{X_a}\not{D}\psi = \frac{n}{4(n-1)(n-2)}\mathcal{R}e_a.\psi - \frac{n}{2(n-2)}P_a.\psi$$

or by defining the Schouten 1-form $K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)}e_a - P_a\right)$, we obtain the second integrability condition for twistor spinors

$$\nabla_{X_a}\not{D}\psi = \frac{n}{2}K_a.\psi \tag{2.5}$$

Moreover, we can define the conformal 2-forms as (for $n > 2$)

$$C_{ab} = R_{ab} - \frac{1}{n-2}\left(P_a \wedge e_b - P_b \wedge e_a\right) + \frac{1}{(n-1)(n-2)}\mathcal{R}e_a \wedge e_b$$

or in terms of Clifford products

$$C_{ab} = R_{ab} - \frac{1}{n-2}(e_a.P_b - e_b.P_a) + \frac{1}{(n-1)(n-2)}\mathcal{R}e_a.e_b$$

By using (2.1), (2.2) and (2.3), one can eailsy found the following third integrability condition for twistor spinors

$$C_{ab}.\psi = 0 \tag{2.6}$$

This means that, if a twistor spinor ψ exists on a manifold M, then it must be in the kernel of the conformal 2-forms C_{ab}.
In conformally-flat manifolds, we have $C_{ab} = 0$ for $n > 2$. So, the third integrability condition is automatically satisfied in conformally-flat manifolds. Hence, conformally-flat manifolds admit twistor spinors. But, they can also exist on nonconformally-flat manifolds with $C_{ab}.\psi = 0$.

iii) Killing spinors

Now, consider a Killing spinor ψ, that is $\nabla_{X_a}\psi = \lambda e_a.\psi$.
By taking the second covariant derivaive and from the definition of the curvature operator, we have

$$\begin{aligned} R(X_a, X_b)\psi &= \nabla_{X_a}\nabla_{X_b}\psi - \nabla_{X_b}\nabla_{X_a}\psi \\ &= -\lambda^2(e_a.e_b - e_b.e_a).\psi \end{aligned}$$

From the identity $e_a.e_b - e_b.e_a = e_a \wedge e_b + i_{X_a}e_b - e_b \wedge e_a - i_{X_b}e_a = 2e_a \wedge e_b$ and the action of the curvature operator on spinors

$$R_{ab}.\psi = -4\lambda^2(e_a \wedge e_b).\psi$$

By Clifford multiplying with e^a from the left, we obtain (from $e^a.R_{ab} = P_b$)

$$\begin{aligned} P_b.\psi &= -4\lambda^2 e^a.(e_a \wedge e_b).\psi \\ &= -4\lambda^2(n-1)e_b.\psi \end{aligned}$$

where we have used $e^a.(e_a \wedge e_b) = e^a \wedge e_a \wedge e_b + i_{X^a}(e_a \wedge e_b) = (i_{X^a}e_a)e_b - e^a \wedge i_{X_a}e_b = (n-1)e_b$ since we have $i_{X^a}e_a = n$ and $e^a \wedge i_{X_a}e_b = e_b$.
Again, by Clifford multiplying with e^b from the left

$$\mathcal{R}\psi = -4\lambda^2 n(n-1)\psi$$

Since the coefficients of ψ in both sides are scalars, we have the identity

$$\mathcal{R} = -4\lambda^2 n(n-1).$$

$\mathcal{R}$ has to be positive or negative (if non-zero), then λ must be real or pure imaginary.
The integrability conditions of Killing spinors, implies that Einstein manifolds ($P_a = ce_a$ with c constant) admit Killing spinors.

IF M is Riemannian, then the existence of Killing spinors requires that M is Einstein.

iv) Parallel spinors

For a parallel spinor ψ, that is $\nabla_X \psi = 0$, the integrability conditions give

$$P_a.\psi = 0.$$

So, Ricci-flat manifolds admit parallel spinors.
In the Riemannian case, the existence of parallel spinors requires that the manifold to be Ricci-flat.

Lecture 3

Holonomy Classification

3.1 Holonomy groups

Let γ be a loop on M, that is

$$\gamma : [0,1] \longrightarrow M \quad \text{and} \quad \gamma(0) = \gamma(1).$$

Take a vector $X \in T_pM$ and parallel transport X via connection ∇ along γ. After a trip along γ, we end up with a new vector $Y \in T_pM$.
Thus, the loop γ and the connection ∇ induce a linear transformation

$$\begin{array}{rcl} g_\gamma : T_pM & \longrightarrow & T_pM \\ X & \longmapsto & Y \end{array}$$

The set of these transformations constitute a group and is called the **holonomy group** at $p \in M$;

$$Hol(p) = \{g_\gamma | \gamma : [0,1] \to M, \gamma(0) = \gamma(1) = p\}.$$

If M is simply connected ($\pi_1(M) = 0$), then $Hol(p)$ is independent of p and γ and we denote the holonomy group of M as $Hol(M)$.
In general, $Hol(M)$ is a subgroup of $GL(n)$.
If we choose the connection ∇ as the Levi-Civita connection, then we have the metric compatibility $\nabla g = 0$ and ∇ preserves the lengths. So, $Hol(M) \subset O(n)$.
If M is orientable, then $Hol(M) \subset SO(n)$.

Holonomy principle: The reduction of the holonomy group $Hol(M)$ of M to a subgroup of $GL(n)$ is equivalent to the existence of a covariantly constant (parallel) section of a bundle E on M.
This parallel section is invariant under the action of $Hol(M)$.

The problem of finding possible holonomy groups of Riemannian and Lorentzian manifolds is solved. However, for arbitrary signature manifolds, it is an unsolved problem.

For the Riemannian case, we have the Berger's table
(A. Besse, *Einstein Manifolds*, 1987);

n	$Hol(M) \subset SO(n)$	Geometry
n	$SO(n)$	generic
$2m$	$U(m)$	Kähler
$2m$	$SU(m)$	Calabi-Yau
$4m$	$Sp(m)$	hyperkähler
$4m$	$Sp(m).Sp(1)$	quaternionic Kähler
7	G_2	exceptional
8	$Spin(7)$	exceptional
16	$Spin(9)$	-

where $Sp(m).Sp(1) = Sp(m) \times Sp(1)/\mathbb{Z}_2$.

Kähler manifolds: For a $n = 2m$ dimensional manifold M, if one can define an almost complex structure $J : TM \longrightarrow TM$ with $J^2 = -1$ and the property $g(JX, JY) = g(X, Y)$ for $X, Y \in TM$, the M is called as an almost complex manifold.
Moreover, if we have $\nabla J = 0$, the J is called as a complex structure and M is called as a complex manifold.
If we can define a symplectic 2-form $\omega(X, Y) = g(JX, Y)$ which is parallel (and hence closed) $\nabla\omega = 0$, then M is a Kähler manifold and ω is called as the Kähler form.
Covariantly constant nature of ω implies the restriction of holonomy to $U(m)$.
One can define (p, q)-forms on M with p holomorphic and q anti-holomorphic components.

Calabi-Yau manifolds: Since $SU(m) \subset U(m)$, manifolds with $SU(m)$ holonomy are Kähler.
Calabi-Yau manifolds are Ricci-flat Kähler manifolds with vanishing first Chern class.
Besides the parallel complex structure J and the parallel Kähler form ω, there is also a parallel complex volume form $\Theta \in \Lambda^{(p,0)}M$.
This implies the restriction of holonomy to $SU(m)$.

hyperkähler manifolds: A hyperkähler structure is a triple $\{I, J, K\}$ of complex (Kähler) structures with the property $IJ = -JI = -K$ and three closed forms ω_i.
Hyperkähler manifolds are Ricci-flat.

quaternionic Kähler manifolds: The structure of quaternionic Kähler manifolds is similar to the hyperkähler case.
However, quaternionic Kähler manifolds are Einstein manifolds.

$\underline{G_2\text{-holonomy manifolds}}$: $G_2 \subset SO(7)$ is the automorphism group of octonions.
There is a parallel 3-form ϕ which is invariant under G_2.
This implies the restriction of holonomy to G_2.
The 4-form $*\phi$ is also parallel.
Mamifolds of G_2-holonomy are Ricci-flat.

$\underline{Spin(7)\text{-holonomy manifolds}}$: $Spin(7) \subset SO(8)$.
There is a parallel 4-form Ψ which is invariant under $Spin(7)$.
This implies the restriction of holonomy to $Spin(7)$.
Manifolds of $Spin(7)$-holonomy are Ricci-flat.

Both parallel forms in G_2 and $Spin(7)$ holonomy manifolds can be constructed from parallel spinors. This is the reason of Ricci-flatness.

3.2 Holonomy classification for parallel and Killing spinors

i) Parallel spinors
We saw that the Ricci-flatness is an integrability condition for the existence of parallel spinors.
Because of the holonomy principle, the existence of parallel spinors also implies the reduction of the holonomy group.
So, in the Riemannian case, parallel spinors exist on Ricci-flat special holonomy manifolds.
Then, from the Berger's table, we obtain Wang's table of manifolds admitting parallel spinors
(M.Y. Wang, Parallel spinors and parallel forms, *Ann. Glob. An. Geom.* 7 (1989) 59);

n	$Hol(M)$	Geometry	parallel spinors
$4m+2$	$SU(m+1)$	Calabi-Yau	$(1,1)$
$4m$	$SU(2m)$	Calabi-Yau	$(2,0)$
$4m$	$Sp(m)$	hyperkähler	$(m+1,0)$
7	G_2	exceptional	1
8	$Spin(7)$	exceptional	$(1,0)$

In even dimensions, complex spinor bundle decomposes into two chiral subbundles $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$.
For $\psi \in \Sigma^+ M$, $iz.\psi = \psi$ and for $\psi \in \Sigma^- M$, $iz.\psi = -\psi$ where z is the volume form. Spinors in $\Sigma^\pm M$ are called Weyl spinors.
(p, q) in the table denotes the number of spinors in $\Sigma^\pm M$ respectively.

Those special holonomy manifolds, have importance in string/M-theory.
In string theory, we have 4 space-time and 6 compact Riemannian dimensions; $M_{10} = M_4 \times M_6$.
In the absence of fluxes, M_4 can be Minkowski space-time. On the other hand, the supersymmetry transformations require the existence of parallel spinors on M_6. Then, M_6 must be a Calabi-Yau ($SU(3)$-holonomy) 3-fold from the table ($m = 1$ in the first row);

$$M_{10} = \underbrace{M_4}_{\text{Minkowski}} \times \underbrace{M_6}_{\text{Calabi-Yau}}$$

In M-theory, we have 4 spce-time dimensions and 7 compact Riemannian dimensions; $M_{11} = M_4 \times M_7$.
If M_4 is Minkowski, then the existence of parallel spinors on M_7 requires that M_7 must be a G_2-holonomy manifold (fourth row in the table);

$$M_{11} = \underbrace{M_4}_{\text{Minkowski}} \times \underbrace{M_7}_{G_2\text{-holonomy}}$$

If we let the decomposition $M_{11} = M_3 \times M_8$, then for M_3 three dimensional Minkowski, M_8 must be a $Spin(7)$-holonomy manifold (fifth row in the table);

$$M_{11} = \underbrace{M_3}_{\text{Minkowski}} \times \underbrace{M_8}_{Spin(7)\text{-holonomy}}$$

Similarly, in F-theory in 12 dimensions, we have

$$M_{12} = \underbrace{M_4}_{\text{Minkowski}} \times \underbrace{M_8}_{Spin(7)\text{-holonomy}}$$

ii) Cone construction
For a manifold (M, g), the warped product manifold $\widetilde{M} = \mathbb{R}^+ \times_{r^2} M$ with the metric $\widetilde{g} = dr^2 + r^2 g$ is called the metric cone of M.
The vector fields $X \in TM$ can be lifted to $T\widetilde{M}$. The basis vector on $T\mathbb{R}^+$ will be denoted by $E = r\frac{\partial}{\partial r}$ (Euler vector).
The Levi-Civita connection $\widetilde{\nabla}$ on $T\widetilde{M}$ is related to the Levi-Civita connection ∇ on TM as follows, for $X, Y \in T\widetilde{M}$

$$\widetilde{\nabla}_E X = \widetilde{\nabla}_X E = X \quad , \quad \widetilde{\nabla}_E E = E$$

$$\widetilde{\nabla}_X Y = \nabla_X Y - g(X, Y)E$$

The relations between curvatures are

$$\widetilde{R}(X, Y)Z = R(X, Y)Z - (g(Y, Z)X - g(X, Z)Y)$$

$$\widetilde{Ric}(X, Y) = Ric(X, Y) - (n-1)g(X, Y)$$

$$\widetilde{\mathcal{R}} = \frac{1}{r}^2 (\mathcal{R} - n(n-1))$$

So, if M is Einstein, then $\widetilde{M}$ is Ricci-flat.
If $\widetilde{M}$ has parallel spinors, then M has Killing spinors corresponding to them.
(This can also be seen from the relations between connection 1-forms $\widetilde{\omega}_{ab}$ and ω_{ab} and the action of connection on spinors $\nabla_X \psi = X(\psi) + \frac{1}{4}\omega_{ab}(X).e^a.e^b.\psi$. In that case, $\widetilde{\nabla}_X \psi$ transforms into $\nabla_X \psi \sim \widetilde{X}.\psi$)

iii) Killing spinors
We have turned the classification problem of manifolds admitting Killing spinors to the classification of manifolds admitting parallel spinors via cone construction.
So, from the Wang's table, we have Bär's table
(C. Bär, Real Killing spinors and holonomy, *Commun. Math. Phys.* 154 (1993) 509);

n	Geometry of M	Cone $\widetilde{M}$	Killing spinors
n	round sphere	flat	$(2^{\lfloor n/2 \rfloor}, 2^{\lfloor n/2 \rfloor})$
$4m-1$	3-Sasaki	hyperkähler	$(m+1, 0)$
$4m-1$	Sasaki-Einstein	Calabi-Yau	$(2, 0)$
$4m+1$	Sasaki-Einstein	Calabi-Yau	$(1, 1)$
6	nearly Kähler	G_2	$(1, 1)$
7	weak G_2	$Spin(7)$	$(1, 0)$

Sasaki-Einstein manifolds: A Sasakian structure on a Riemannian manifold M is a KIlling vector field K of unit norm with the property

$$\nabla_X \nabla_Y K = -g(X,Y)K + \widetilde{K}(Y)X \quad \text{for } X, Y \in TM$$

If a Sasakian manifold is also Einstein, then we have a Sasaki-Einstin manifold.

3-Sasaki manifolds: A 3-Sasakian structure on a Riemannian manifold M is a triple $K_i, i = 1,2,3$ of Sasakian structures with the relations

$$\nabla_{K_i} K_j = \epsilon_{ijk} K_k$$

and some extra properties.

nearly Kähler manifolds: A nearly Kähler structure on a Riemannian manifold M is an almost complex structure $J : TM \to TM$ with the property

$$i_X \nabla_X J = 0$$

but $\nabla J \neq 0$. So, it is not Kähler.

weak G_2 manifolds: On a weak G_2 manifold, there is a 3-form ϕ with the property

$$d\phi = \lambda * \phi$$

So, it is not closed as in the G_2 case. Indeed, ϕ is a Killing-Yano form.

The importance in string/M-theory;
In string theory, we have $M_{10} = M_4 \times M_6$. In the presence of fluxes, M_4 can be Anti-de Sitter (AdS) and the supersymmetry transformations require the existence of Killing spinors on M_6. Then, M_6 must be nearly Kähler;

$$M_{10} = \underbrace{M_4}_{AdS_4} \times \underbrace{M_6}_{\text{nearly Kähler}}$$

If we let $M_{10} = M_5 \times M_5$, then we have

$$M_{10} = \underbrace{M_5}_{AdS_5} \times \underbrace{M_5}_{S^5} \qquad \text{or} \qquad M_{10} = \underbrace{M_5}_{AdS_5} \times \underbrace{M_5}_{\text{Sasaki-Einstein}}$$

In M-theory, we have $M_{11} = M_4 \times M_7$. In the presence of fluxes, M_4 is AdS and if there is no internal flux, then M_7 is S^7. However, in the presence of internal fluxes, the existence of Killing spinors requires that M_7 is weak G_2-holonomy manifold;

$$M_{11} = \underbrace{M_4}_{AdS_4} \times \underbrace{M_7}_{S^7} \qquad \text{or} \qquad M_{11} = \underbrace{M_4}_{AdS_4} \times \underbrace{M_7}_{\text{weak } G_2}$$

Similarly, if we let $M_{11} = M_5 \times M_6$, then we have

$$M_{11} = \underbrace{M_5}_{AdS_5} \times \underbrace{M_6}_{S^6} \qquad \text{or} \qquad M_{11} = \underbrace{M_5}_{AdS_5} \times \underbrace{M_6}_{\text{nearly Kähler}}$$

iv) For the Lorentizan case, the manifolds admitting parallel spinors are either Ricci-flat $\times \mathbb{R}^+$ or Brinkmann spaces (including pp-wave space-times).

Lecture 4

Spinor Bilinears

4.1 Dirac currents

We can define a Spin-invariant inner product on spinor fields.
For $\psi, \phi \in \Sigma M$, we denote the inner product as

$$(\psi, \phi) = \pm(\phi, \psi)^j$$

where j is an involution in $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ (depending on the Clifford algebra in relevant dimension). So, j is identity in $\mathbb{R}$, identity or complex conjugation $({}^*)$ in $\mathbb{C}$, quaternionic conjugation or reversion in $\mathbb{H}$.
The inner product can be symmetric or anti-symmetric (symplectic) depending on the dimension.
For any $\alpha \in Cl(M)$ and $c \in \mathbb{D}$, we have

$$\begin{aligned} (\psi, \alpha.\phi) &= (\alpha^{\mathcal{J}}.\psi, \phi) \\ (c\psi, \phi) &= c^j(\psi, \phi) \end{aligned}$$

where $\mathcal{J}$ can be ξ, $\xi\eta$, ξ^* or $\xi\eta^*$ depending on the dimension.
From this inner product, we can define the space of dual spinors $\Sigma^* M$.
For $\overline{\psi} \in \Sigma^* M$, we have

$$\overline{\psi}(\phi) = (\psi, \phi)$$

Let us consider the tensor product $\Sigma M \otimes \Sigma^* M$. Its action on ΣM is given by

$$(\psi \otimes \overline{\phi})\kappa = (\phi, \kappa)\psi$$

for $\psi, \kappa \in \Sigma M$ and $\overline{\phi} \in \Sigma^* M$.
So, the elements of $\Sigma M \otimes \Sigma^* M$ are linear transformations on ΣM.
It is isomorphic to $Cl(M)$ (to its simple part), since we have $c : Cl(M) \otimes \Sigma M \to \Sigma M$.
Then, we can write $\psi \otimes \overline{\phi} \in \Sigma M \otimes \Sigma^* M$ in terms of differential forms which are elements of $Cl(M)$.

For any orthonormal co-frame basis $\{e^a\}$, we have the Fierz identity in terms of the inner product

$$\begin{aligned}\psi\overline{\phi} &= (\phi,\psi) + (\phi, e_a.\psi)e^a + (\phi, e_{ba}.\psi)e^{ab} + ... + \\ &\quad +(\phi, e_{a_p...a_2a_1}.\psi)e^{a_1a_2...a_p} + ... + (-1)^{\lfloor n/2 \rfloor}(\phi, z.\psi)z\end{aligned}$$

where $e^{a_1a_2...a_p} = e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_p}$ and z is the volume form.
We can define the p-form projections of $\psi\overline{\phi}$ as $(\psi\overline{\phi})_p$ which are called **spinor bilinears**.

If we take $\phi = \psi$, we have the following definitions;
Dirac current: The vector field V_ψ which is the metric dual of the 1-form $(\psi\overline{\psi})_1$.
In general, the map

$$\begin{aligned}(\quad)_1 : \Sigma M &\longrightarrow T^*M \cong TM \\ \psi &\longmapsto (\psi\overline{\psi})_1 \cong V_\psi\end{aligned}$$

is called the squaring map of a spinor.
p-form Dirac current: The p-form component of $\psi\overline{\psi}$;

$$(\psi\overline{\psi})_p = (\psi, e_{a_p...a_2a_1}.\psi)e^{a_1a_2...a_p}$$

4.2 CKY and KY forms

We investigate the properties of p-form Dirac currents for special spinors.

i) Twistor spinors

The Levi-Civita connection ∇ is compatible with the spinor inner product and the duality operation. Namely, we have

$$\begin{aligned}\nabla_X(\psi,\phi) &= (\nabla_X\psi,\phi) + (\psi,\nabla_X\phi) \\ \nabla_X\overline{\psi} &= \overline{\nabla_X\psi}\end{aligned}$$

for $X \in TM$, $\psi,\phi \in \Sigma M$.

For a twistor spinor ψ, we can calculate the covariant derivative of the p-form Dirac current $(\psi\overline{\psi})_p$ as

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_p &= \big((\nabla_{X_a}\psi)\overline{\psi}\big)_p + \big(\psi(\overline{\nabla_{X_a}\psi})\big)_p \\ &= \frac{1}{n}\big((e_a.\not{D}\psi)\overline{\psi}\big)_p + \frac{1}{n}\big(\psi(\overline{e_a.\not{D}\psi})\big)_p\end{aligned}$$

where we have used the twistor equation $\nabla_{X_a}\psi = \frac{1}{n}e_a.\not{D}\psi$.
From the following properties of the tensor products of spinors and dual spinors, for $\alpha \in Cl(M)$, $\psi \in \Sigma M$ and $\overline{\phi} \in \Sigma^*M$

$$\begin{aligned}\alpha.(\psi\otimes\overline{\phi}) &= \alpha.\psi\otimes\overline{\phi} \\ (\psi\otimes\overline{\phi}).\alpha &= \psi\otimes\overline{\alpha^{\mathcal{J}}.\phi}\end{aligned}$$

we can write $\psi(\overline{e_a.\not{D}\psi}) = (\psi\overline{\not{D}\psi}).e_a^{\mathcal{J}}$. The definition of the Dirac operator on spinors $\not{D} = e^b.\nabla_{X_b}$ and the property $e^{b\mathcal{J}}.e_a{}^{\mathcal{J}} = e^b.e_a$ for $\mathcal{J} = \xi$ or $\xi\eta$ gives

$$\nabla_{X_a}(\psi\overline{\psi})_p = \frac{1}{n}\left(e_a.e^b.(\nabla_{X_b}\psi)\overline{\psi} + \psi(\overline{\nabla_{X_b}\psi}).e^b.e_a\right)_p$$

Since ∇ is compatible with the tensor product, we have $(\nabla_{X_b}\psi)\overline{\psi} = \nabla_{X_b}(\psi\overline{\psi}) - \psi(\overline{\nabla_{X_b}\psi})$. So, we can write

$$\nabla_{X_a}(\psi\overline{\psi})_p = \frac{1}{n}\left(e_a.e^b.\nabla_{X_b}(\psi\overline{\psi}) - e_a.e^b.\psi(\overline{\nabla_{X_b}\psi}) + \psi(\overline{\nabla_{X_b}\psi}).e^b.e_a\right)_p$$

Now, we analyze each term on the right hand side of the above equation. For the first term, we can use the definition of Hodge-de Rham operator $\not{d} = e^a.\nabla_{X_a}$ on differential forms which are elements of $Cl(M)$ and we can write

$$e_a.e^b.\nabla_{X_b}(\psi\overline{\psi}) = e_a.\not{d}(\psi\overline{\psi})$$

For the second and third terms, we can use the explicit expansion of Clifford product in terms of wedge product and interior product as $e_a.\alpha = e_a \wedge \alpha + i_{X_a}\alpha$ and $\alpha.e_a = e_a \wedge \eta\alpha - i_{X_a}\eta\alpha$. So, we have

$$\begin{aligned} e_a.e^b.\psi(\overline{\nabla_{X_b}\psi}) &= e_a \wedge e^b \wedge \psi(\overline{\nabla_{X_b}\psi}) - i_{X_a}\left(e^b \wedge \psi(\overline{\nabla_{X_b}\psi})\right) \\ &\quad + e_a \wedge i_{X^b}\left(\psi(\overline{\nabla_{X_b}\psi})\right) + i_{X_a}i_{X^b}\left(\psi(\overline{\nabla_{X_b}\psi})\right) \end{aligned}$$

and

$$\begin{aligned} \psi(\overline{\nabla_{X_b}\psi}).e^b.e_a &= e_a \wedge \eta\left(e^b \wedge \eta\left(\psi(\overline{\nabla_{X_b}\psi})\right)\right) - i_{X_a}\eta\left(e^b \wedge \eta\left(\psi(\overline{\nabla_{X_b}\psi})\right)\right) \\ &\quad - e_a \wedge \eta i_{X^b}\eta\left(\psi(\overline{\nabla_{X_b}\psi})\right) + i_{X_a}\eta i_{X^b}\eta\left(\psi(\overline{\nabla_{X_b}\psi})\right) \end{aligned}$$

By summing all the terms and considering the degrees of differential forms, we obtain

$$\nabla_{X_a}(\psi\overline{\psi})_p = \frac{1}{n}\left[\left(e_a.\not{d}(\psi\overline{\psi})\right)_p - 2e_a \wedge e^b \wedge \left(\psi(\overline{\nabla_{X_b}\psi})\right)_{p-2} - 2i_{X_a}i_{X^b}\left(\psi(\overline{\nabla_{X_b}\psi})\right)_{p+2}\right]$$

By using $\left(e_a.\not{d}(\psi\overline{\psi})\right)_p = e_a \wedge \left(\not{d}(\psi\overline{\psi})\right)_{p-1} + i_{X_a}\left(\not{d}(\psi\overline{\psi})\right)_{p+1}$, we find

$$\begin{aligned} \nabla_{X_a}(\psi\overline{\psi})_p &= \frac{1}{n}\Big[e_a \wedge \left(\not{d}(\psi\overline{\psi}) - 2e^b \wedge \left(\psi(\overline{\nabla_{X_b}\psi})\right)\right)_{p-1} \\ &\quad + i_{X_a}\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}\left(\psi(\overline{\nabla_{X_b}\psi})\right)\right)_{p+1}\Big] \end{aligned} \tag{4.1}$$

By taking the wedge product of (4.1) with $e^a\wedge$ from the left and using $e^a \wedge \nabla_{X_a} = d$ and $e^a \wedge i_{X_a}\alpha = p\alpha$ for a p-form α

$$d(\psi\overline{\psi})_p = \frac{p+1}{n}\left(\not{d}(\psi\overline{\psi}) - 2i_{X^b}\left(\psi(\overline{\nabla_{X_b}\psi})\right)\right)_{p+1}$$

By taking the interior derivative i_{X^a} of (4.1) and using $-i_{X^a}\nabla_{X_a} = \delta$

$$\delta(\psi\overline{\psi})_p = -\frac{n-p+1}{n}\left(\not{d}(\psi\overline{\psi}) - 2e^b \wedge \left(\psi(\overline{\nabla_{X_b}\psi})\right)\right)_{p-1}$$

So, by comparing the last three equations, we obtain

$$\nabla_{X_a}(\psi\overline{\psi})_p = \frac{1}{p+1}i_{X_a}d(\psi\overline{\psi})_p - \frac{1}{n-p+1}e_a \wedge \delta(\psi\overline{\psi})_p$$

This equation has a special meaning. If a p-form ω satisfies the equation

$$\nabla_X\omega = \frac{1}{p+1}i_X d\omega - \frac{1}{n-p+1}\widetilde{X} \wedge \delta\omega$$

for any $X \in TM$, then ω is called as a **conformal Killing-Yano** (CKY) p-form.
So, p-form Dirac currents of twistor spinors are CKY forms.
CKY forms are antisymmetric generalizations of conformal Killing vector fields to higher degree forms.
For $p = 1$, ω is metric dual of a conformal Killing vector.

ii) Killing spinors
For a Killing spinor ψ, the covariant derivative of $(\psi\overline{\psi})_p$ gives

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_p &= \left((\nabla_{X_a}\psi)\overline{\psi}\right)_p + \left(\psi(\overline{\nabla_{X_a}\psi})\right)_p \\ &= \left(\lambda e_a.\psi\overline{\psi}\right)_p + \left(\psi(\overline{\lambda e_a.\psi})\right)_p\end{aligned}$$

where we have used the Killing spinor equation $\nabla_{X_a}\psi = \lambda e_a.\psi$.
From the properties of the inner product, we have $\psi(\overline{\lambda e_a.\psi}) = \lambda^j(\psi\overline{\psi})e_a^{\mathcal{J}}$.
So, we can write

$$\begin{aligned}\nabla_{X_a}(\psi\overline{\psi})_p &= \left(\lambda e_a.\psi\overline{\psi}\right)_p + \left(\lambda^j(\psi\overline{\psi})e_a^{\mathcal{J}}\right)_p \\ &= \lambda e_a \wedge (\psi\overline{\psi})_{p-1} + \lambda i_{X_a}(\psi\overline{\psi})_{p+1} + \lambda^j e_a^{\mathcal{J}} \wedge (\psi\overline{\psi})_{p-1}^{\eta} - \lambda^j i_{\widetilde{e_a^{\mathcal{J}}}}(\psi\overline{\psi})_{p+1}^{\eta}\end{aligned}$$

where we have used the expansion of the Clifford product in terms of the wedge product and interior derivative.
By taking wedge product with e^a from the left

$$d(\psi\overline{\psi})_p = \lambda(p+1)(\psi\overline{\psi})_{p+1} - \lambda^j\mathrm{sgn}(e_a^{\mathcal{J}})(p+1)(\psi\overline{\psi})_{p+1}^{\eta}$$

and by taking interior derivative with i_{X^a}

$$\delta(\psi\overline{\psi})_p = -\lambda(n-p+1)(\psi\overline{\psi})_{p-1} - \lambda^j\mathrm{sgn}(e_a^{\mathcal{J}})(n-p+1)(\psi\overline{\psi})_{p-1}^{\eta}$$

Here, we have four parameters to choose; λ real or pure imaginary, $j = Id$ or $*$, $\mathcal{J} = \xi, \xi^*, \xi\eta$ or $\xi\eta^*$ and p even or odd (we consider complex spinors).
So, there are 16 possibilities to choose;

λ	j	$\mathcal{J}$	p	
Re	Id	ξ	odd	even
Re	Id	ξ^*	odd	even
Re	$*$	ξ	odd	even
Re	$*$	ξ^*	odd	even
Im	Id	ξ	odd	even
Im	Id	ξ^*	odd	even
Im	$*$	ξ	even	odd
Im	$*$	ξ^*	even	odd
Re	Id	$\xi\eta$	even	odd
Re	Id	$\xi\eta^*$	even	odd
Re	$*$	$\xi\eta$	even	odd
Re	$*$	$\xi\eta^*$	even	odd
Im	Id	$\xi\eta$	even	odd
Im	Id	$\xi\eta^*$	even	odd
Im	$*$	$\xi\eta$	odd	even
Im	$*$	$\xi\eta$	odd	even

By the detailed considerations of the possibilities, one can see that we have two different cases. The first column of p gives Case 1 and the second column of p gives Case 2 in which the equations above transforms into the following equalities.

Case 1:

$$\begin{aligned}
\nabla_{X_a}(\psi\overline{\psi})_p &= 2\lambda e_a \wedge (\psi\overline{\psi})_{p-1} \\
d(\psi\overline{\psi})_p &= 0 \\
\delta(\psi\overline{\psi})_p &= -2\lambda(n-p+1)(\psi\overline{\psi})_{p-1}
\end{aligned}$$

Case 2:

$$\begin{aligned}
\nabla_{X_a}(\psi\overline{\psi})_p &= 2\lambda i_{X_a}(\psi\overline{\psi})_{p+1} \\
d(\psi\overline{\psi})_p &= 2\lambda(p+1)(\psi\overline{\psi})_{p+1} \\
\delta(\psi\overline{\psi})_p &= 0
\end{aligned}$$

By comparing the equations in Case 1, we find

$$\nabla_{X_a}(\psi\overline{\psi})_p = -\frac{1}{n-p+1} e_a \wedge \delta(\psi\overline{\psi})_p$$

and in Case 2, we have

$$\nabla_{X_a}(\psi\overline{\psi})_p = \frac{1}{p+1} i_{X_a} d(\psi\overline{\psi})_p$$

By comparing these equalities with the CKY equation, one can see that Case 1 and Case 2 correspond to two parts of the CKY equation;

$$\nabla_X\omega = \underbrace{\frac{1}{p+1}i_X d\omega}_{\text{KY part}} - \underbrace{\frac{1}{n-p+1}\widetilde{X}\wedge\delta\omega}_{\text{CCKY part}}$$

Case 1 corresponds to the CCKY part and Case 2 corresponds to the KY part.
Killing-Yano (KY) forms are co-closed ($\delta\omega = 0$) CKY forms and satisfy

$$\nabla_X\omega = \frac{1}{p+1}i_X d\omega$$

They are antisymmetric generalizations of Killing vector fields to higher degree forms.
For $p = 1$, ω is the metric dual of a Killing vector.
Closed CKY (CCKY) forms are CKY forms with $d\omega = 0$ and satisfy

$$\nabla_X\omega = -\frac{1}{n-p+1}\widetilde{X}\wedge\delta\omega$$

CKY equation has Hodge duality invariance, namely if ω is a CKY p-form, then $*\omega$ is also a CKY $(n-p)$-form. To see this, let us apply the Hodge star operator * to the CKY equation

$$*\nabla_X\omega = \frac{1}{p+1} * i_X d\omega - \frac{1}{n-p+1} * (\widetilde{X}\wedge\delta\omega)$$

Since ∇ is metric compatible, we have $*\nabla_X = \nabla_X *$. From the definition $\delta = *^{-1}d*\eta$ and the identity $*(\alpha\wedge\widetilde{X}) = i_X * \alpha$ for any p-form α, we can write

$$\begin{aligned}
*i_X d\omega &= *i_X **^{-1}d**^{-1}\omega \\
&= **(\delta\eta *^{-1}\omega\widetilde{X}) \\
&= **(-\widetilde{X}\wedge\delta *^{-1}\omega) \\
&= **^{-1}(-\widetilde{X}\wedge\delta*\omega) \\
&= -\widetilde{X}\wedge\delta*\omega
\end{aligned}$$

where we have used that $*^{-1}$ is proportional to * from the identity $**\alpha = (-1)^{p(n-p)}\frac{detg}{|detg|}\alpha$. We can also write

$$\begin{aligned}
*(\widetilde{X}\wedge\delta\omega) &= *(-\delta\eta\omega\wedge\widetilde{X}) \\
&= -i_X * \delta\eta\omega \\
&= -i_X **^{-1}d*\eta\eta\omega \\
&= -i_X d*\omega
\end{aligned}$$

So, we obtain

$$\nabla_X *\omega = \underbrace{\frac{1}{n-p+1}i_X d*\omega}_{\text{*KY part}} - \underbrace{\frac{1}{p+1}\widetilde{X}\wedge\delta*\omega}_{\text{*CCKY part}}$$

However, as can be seen from the above analysis, the two parts of the CKY equation transform into each other under *. Namely, we have

$$\text{KY} \longrightarrow {}^*\text{CCKY} \quad \text{and} \quad \text{CCKY} \longrightarrow {}^*\text{KY}$$

This means that CCKY forms are Hodge duals of KY forms.
Then, p-form Dirac currents of Killing spinors correspond to KY forms (in Case 2) or the Hodge duals of p-form Dirac cuurents of Killing spinors correspod to KY forms (in Case 1) depending on the inner product in relevant dimension;
Case 1:

$$\nabla_{X_a} * (\psi\overline{\psi})_p = \frac{1}{n-p+1} i_{X_a} d * (\psi\overline{\psi})_p$$

Case 2:

$$\nabla_{X_a} (\psi\overline{\psi})_p = \frac{1}{p+1} i_{X_a} d(\psi\overline{\psi})_p$$

The equations satisfied by the p-form Dirac currents of Killing spinors have an analogous structure with (generalized) Maxwell equations

$$dF = 0 \quad , \quad d * F = j$$

In Case 1, we have

$$\begin{aligned} d(\psi\overline{\psi})_p &= 0 \\ d * (\psi\overline{\psi})_p &= -2(-1)^p \lambda (n-p+1) * (\psi\overline{\psi})_{p-1} \end{aligned}$$

Hence, p-form Dirac currents $(\psi\overline{\psi})_p$ behave like the field strength F of the p-form Maxwell field and the source term j is constructed from the Hodge duals of the one lower degree Dirac currents $*(\psi\overline{\psi})_{p-1}$.
In Case 2, we have

$$\begin{aligned} d(\psi\overline{\psi})_p &= 2\lambda(p+1)(\psi\overline{\psi})_{p+1} \\ d * (\psi\overline{\psi})_p &= 0 \end{aligned}$$

So, $*(\psi\overline{\psi})_p$ behave like the $(n-p)$-form Maxwell field strength F and source term j is the Dirac cuurent of one higher degree $(\psi\overline{\psi})_{p+1}$.

Two sets of equations in Case 1 and Case 2 have also interesting relations with Duffin-Kemmer-Petiau (DKP) equations. DKP equations are first-order equations describing integer spin particles and are written as

$$d\phi_+ - \delta\phi_- = \mu\phi$$

where $\phi \in Cl(M)$ is the integer spin field which can be written as a sum of even and odd parts $\phi = \phi_+ + \phi_-$ with $\phi_\pm = \frac{1}{2}(1 \pm \eta)\phi$ and μ is the mass.
Between Case 1 and Case 2, the parity (oddness or evenness) of p changes. So, if $(\psi\overline{\psi})_p$ satisfies Case 1, then $(\psi\overline{\psi})_{p-1}$ satisfies Case 2. Hence, we have

$$\begin{aligned} d(\psi\overline{\psi})_{p-1} = 2\lambda p(\psi\overline{\psi})_p \quad &, \quad \delta(\psi\overline{\psi})_{p-1} = 0 \\ \delta(\psi\overline{\psi})_p = -2\lambda(n-p+1)(\psi\overline{\psi})_{p-1} \quad &, \quad d(\psi\overline{\psi})_p = 0 \end{aligned}$$

If we choose $p = \frac{n+1}{2}$ and define $\phi_\pm = (\psi\overline{\psi})_{p-1}$, $\phi_\mp(\psi\overline{\psi})_p$ for p odd or even with $\mu = 2\lambda p$, we obtain

$$\begin{aligned} d\phi_\pm = \mu\phi_\mp \quad &, \quad \delta\phi_\pm = 0 \\ \delta\phi_\mp = -\mu\phi_\pm \quad &, \quad d\phi_\mp = 0 \end{aligned}$$

These are equivalent to the DKP equations.

Moreover, p-form Dirac currents of Killing spinors are eigenforms of the Laplace-Beltrami operator $\Delta = -d\delta - \delta d$.

If $(\psi\overline{\psi})_p$ satisfies Case 1, then $(\psi\overline{\psi})_{p-1}$ satisfies Case 2 and we have

$$\begin{aligned} \Delta(\psi\overline{\psi})_p = -d\delta(\psi\overline{\psi})_p &= 2\lambda(n-p+1)d(\psi\overline{\psi})_{p-1} \\ &= 4\lambda^2 p(n-p+1)(\psi\overline{\psi})_p \end{aligned}$$

If $(\psi\overline{\psi})_p$ satisfies Case 2, then $(\psi\overline{\psi})_{p+1}$ satisfies Case 1 and we have

$$\begin{aligned} \Delta(\psi\overline{\psi})_p = -\delta d(\psi\overline{\psi})_p &= -2\lambda(p+1)\delta(\psi\overline{\psi})_{p+1} \\ &= 4\lambda^2(p+1)(n-p+2)(\psi\overline{\psi})_p \end{aligned}$$

iii) Parallel spinors

If ψ is a parallel spinor $\nabla_X\psi = 0$, then p-form Dirac currents are parallel forms

$$\nabla_X(\psi\overline{\psi})_p = 0$$

Indeed, they are harmonic forms which are in the kernel of Δ

$$\Delta(\psi\overline{\psi})_p = 0$$

More details can be found in

Ö. Açık and Ü. Ertem, Higher-degree Dirac currents of twistor and Killing spinors in supergravity theories, *Class. Quant. Grav.* 32 (2015) 175007.

Lecture 5

Symmetry Operators

5.1 Lie derivatives

For two vector fields $X, Y \in TM$, the Lie derivative of Y with respect to X which is denoted by $\mathcal{L}_X Y$ is the change of Y with respect to the flow of X.
It can be written as

$$\mathcal{L}_X Y = [X, Y] = X(Y) - Y(X).$$

Lie derivative has the property

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}.$$

For a differential form $\alpha \in \Lambda M$, the Lie derivative $\mathcal{L}_X$ with respect to X can be written in terms of d and i_X as

$$\mathcal{L}_X \alpha = i_X d\alpha + d i_X \alpha.$$

We consider Lie derivatives with respect to Killing and conformal Killing vector fields. They are generators of isometries and conformal isometries

$$\begin{aligned} \mathcal{L}_K g = 0 \qquad & ; \qquad K \text{ Killing vector} \\ \mathcal{L}_C g = 2\mu g \qquad & ; \qquad C \text{ conformal Killing vector} \quad (\mu \text{ function}) \end{aligned}$$

If $\alpha \in Cl(M)$, namely an inhomogeneous differential form, then the Lie derivative w.r.t. a Killing vector K is written as

$$\mathcal{L}_K \alpha = \nabla_K \alpha + \frac{1}{4}[d\widetilde{K}, \alpha]_{Cl}$$

where $\widetilde{K}$ is the metric dual of K and $[\,]_{Cl}$ is the Clifford bracket which is defined for $\alpha, \beta \in Cl(M)$ as $[\alpha, \beta]_{Cl} = \alpha.\beta - \beta.\alpha$.
For a spinor field $\psi \in \Sigma M$, the Lie derivative w.r.t. K is

$$\pounds_K \psi = \nabla_K \psi + \frac{1}{4} d\widetilde{K}.\psi$$

It has a derivative property on spinors $\phi \in \Sigma M$, $\overline{\psi} \in \Sigma^* M$

$$\pounds_K(\phi\overline{\psi}) = (\pounds_K\phi)\overline{\psi} + \phi(\overline{\pounds_K\psi})$$

and compatible with the spinor duality operation. This is also true for conformal Killing vectors C, but not true for an arbitrary vector field X.
So, the Lie derivative on spinors is only defined w.r.t. isometries.

5.2 Symmetry operators

A symmetry operator is an operator that maps a solution of an equation to another solution.
We consider first-order symmetry operators for spinor field equations.

i) Dirac equation
For the massive Dirac equation $\not{D}\psi = m\psi$, the Lie derivative w.r.t. a Killing vector K is a symmetry operator.
So, if ψ is a solution, then $\pounds_K\psi$ is also a solution

$$\not{D}\pounds_K\psi = m\pounds_K\psi.$$

For the massless Dirac equation $\not{D}\psi = 0$, the operator $\pounds_C + \frac{1}{2}(n-1)\mu$ defined w.r.t. a conformal Killing vector C ($\mathcal{L}_C g = 2\mu g$) is a symmetry operator in n dimensions, namely we have

$$\not{D}\left(\pounds_C - \frac{1}{2}(n-1)\mu\right)\psi = 0.$$

These can be generalized to higher-degree KY and CKY forms.
For the massive Dirac equation, the following operator written in terms of a KY p-form ω (for p odd)

$$L_\omega\psi = (i_{X^a}\omega).\nabla_{X_a}\psi + \frac{p}{2(p+1)}d\omega.\psi$$

is a symmetry operator which reduces to $\pounds_K$ for $p = 1$. (For p even, $L_\omega z$ is a symmetry operator with z is the volume form).
For the massless Dirac equation, the operator written in terms of a CKY p-form ω

$$L_\omega\psi = (i_{X^a}\omega).\nabla_{x_a}\psi + \frac{p}{2(p+1)}d\omega.\psi - \frac{n-p}{2(n-p+1)}\delta\omega.\psi$$

is a symmetry operator which reduces to $\pounds_C - \frac{1}{2}(n-1)\mu$ for $p = 1$ (since $\delta\widetilde{C} = -n\mu$).

ii) Killing spinors
Killing spinors are solutions of the massive Dirac equation wihch are also twistor spinors.

However, this does not mean that the symmetry operators of the massive Dirac equation must preserve the subset of Killing spinors.
For a Killing vector K, the Lie derivative $\pounds_K$ is a symmetry operator for the Killing spinor equation;

$$\nabla_X \pounds_K \psi = \lambda \widetilde{X}.\pounds_K \psi.$$

For higher-degree KY forms ω, we investigate that in which circumstances the operator L_ω defined for the massive Dirac equation is also a symmetry operator for Killing spinors;

$$\nabla_X L_\omega \psi = \lambda \widetilde{X}.L_\omega \psi.$$

For a Killing spinor ψ and a KY p-form ω, we can write

$$\begin{aligned} L_\omega \psi &= (i_{X^a}\omega).\nabla_{X_a}\psi + \frac{p}{2(p+1)}d\omega.\psi \\ &= \lambda(i_{X^a}\omega).e_a.\psi + \frac{p}{2(p+1)}d\omega.\psi \\ &= (-1)^{p-1}\lambda p\omega.\psi + \frac{p}{2(p+1)}d\omega.\psi \end{aligned}$$

where we have used the Killing spinor equation and the identities $(i_{X^a}\omega).e_a = e_a \wedge \eta(i_{X^a}\omega) - i_{X_a}\eta i_{X^a}\omega$ (second term is zero since $i_X i_Y$ is antisymmetric) and $e_a \wedge i_{X^a}\omega = p\omega$.
If we take p odd, then

$$L_\omega \psi = \lambda p\omega.\psi + \frac{p}{2(p+1)}d\omega.\psi$$

By calculating the covariant derivative, we obtain

$$\begin{aligned} \nabla_{X_a} L_\omega \psi &= \nabla_{X_a}\left(\lambda p\omega.\psi + \frac{p}{2(p+1)}d\omega.\psi\right) \\ &= \lambda p\nabla_{X_a}\omega.\psi + \lambda p\omega.\nabla_{X_a}\psi + \frac{p}{2(p+1)}\nabla_{X_a}d\omega.\psi + \frac{p}{2(p+1)}d\omega.\nabla_{X_a}\psi \end{aligned}$$

We can use the Killing spinor equation $\nabla_{X_a}\psi = \lambda e_a.\psi$, KY equation $\nabla_{X_a}\omega = \frac{1}{p+1}i_{X_a}d\omega$ and the integrability consition for the KY equation which can be calculated as

$$\begin{aligned} \nabla_{X_b}\nabla_{X_a}\omega &= \frac{1}{p+1}\nabla_{X_b}i_{X_a}d\omega \\ &= \frac{1}{p+1}i_{X_a}\nabla_{X_b}d\omega \end{aligned}$$

from $[i_{X_a}, \nabla_{X_b}] = 0$ in normal coordinates (in general, we have $[\nabla_X, i_Y] = i_{\nabla_X Y}$).
By using the curvature operator $R(X_a, X_b) = \nabla_{X_a}\nabla_{X_b} - \nabla_{X_b}\nabla_{X_a}$, we can write

$$R(X_a, X_b)\omega = \frac{1}{p+1}\left(i_{X_b}\nabla_{X_a} - i_{X_a}\nabla_{X_b}\right)d\omega.$$

By multiplying $e^a\wedge$ from the left, we have

$$\begin{aligned}
e^a \wedge R(X_a, X_b)\omega &= \frac{1}{p+1}\left(e^a \wedge i_{X_b}\nabla_{X_a} d\omega - e^a \wedge i_{X_a}\nabla_{X_b} d\omega\right) \\
&= \frac{1}{p+1}\left(e^a \wedge \nabla_{X_a} i_{X_b} d\omega - (p+1)\nabla_{X_b} d\omega\right)
\end{aligned}$$

where we have used $[i_{X_b}, \nabla_{X_a}] = 0$ and $e^a \wedge i_{X_a}\nabla_{X_b} d\omega = (p+1)\nabla_{X_b} d\omega$. From the identities $e^a \wedge \nabla_{X_a} = d$, $d^2 = 0$ and $\nabla_{X_b} = i_{X_b} d + d i_{X_b}$ in normal coordinates, we can write

$$e^a \wedge R(X_a, X_b)\omega = -\frac{p}{p+1}\nabla_{X_b} d\omega.$$

The action of the curvature operator on a p-form ω is

$$R(X_a, X_b)\omega = -i_{X^c} R_{ab} \wedge i_{X_c}\omega$$

(for proof, see I.M. Benn and R.W. Tucker, *An Introduction to Spinors and Geometry with Applications in Physics*, 1987).
So, we have

$$\begin{aligned}
e^a \wedge R(X_a, X_b)\omega &= -e^a \wedge i_{X^c} R_{ab} \wedge i_{X_c}\omega \\
&= (i_{X^c}(e^a \wedge R_{ab} - R^c_b)) \wedge i_{X_c}\omega \\
&= -R_{cb} \wedge i_{X^c}\omega.
\end{aligned}$$

Then, we obtain the integrability condition for KY forms ω as follows

$$\nabla_{X_a} d\omega = \frac{p+1}{p} R_{ba} \wedge i_{X^b}\omega.$$

Now, we can write the covariant derivative of $L_\omega\psi$ as

$$\begin{aligned}
\nabla_{X_a} L_\omega \psi &= \lambda \frac{p}{p+1} i_{X_a} d\omega.\psi + \lambda^2 p\omega.e_a.\psi \\
&\quad + \frac{1}{2}(R_{ba} \wedge i_{X^b}\omega).\psi + \lambda \frac{p}{2(p+1)} d\omega.e_a.\psi.
\end{aligned}$$

On the other hand, for the right hand side of the Killing spinor equation, we have

$$\lambda e_a.L_\omega\psi = \lambda^2 p e_a.\omega.\psi + \lambda \frac{p}{2(p+1)} e_a.d\omega.\psi.$$

Obviously, the right hand sides of the last two equalities are not equal to each other.
Let us consider special KY forms which have the following special integrability condition

$$\nabla_X d\omega = -c(p+1)\widetilde{X} \wedge \omega$$

where c is a constant. For example, in constant curvature manfiolds we have $R_{ab} = c e_a \wedge e_b$ and the special integrability condition is satified by all KY forms since the right hand side of the ordinary integrability condition gives

$$\begin{aligned} \frac{p}{p+1}R_{ba}\wedge i_{X^b}\omega &= c\frac{p+1}{p}e_b\wedge e_a\wedge i_{X^b}\omega \\ &= -c(p+1)e_a\wedge\omega \end{aligned}$$

where we have used $e_b\wedge e_a=-e_a\wedge e_b$ and $e_b\wedge i_{X^b}\omega=p\omega$.
So, all KY forms are special KY forms in constant curvature manifolds.
If $R_{ab}=ce_a\wedge e_b$, then $P_b=i_{X^a}R_{ab}=c(n-1)e_b$ and $\mathcal{R}=i_{X^b}P_b=cn(n-1)$, so we have

$$c=\frac{\mathcal{R}}{n(n-1)}\qquad\text{and}\qquad R_{ab}=\frac{\mathcal{R}}{n(n-1)}e_a\wedge e_b.$$

Moreover, we know from the integrability condition of Killing spinors that $\mathcal{R}=-4\lambda^2n(n-1)$ which implies that

$$c=-4\lambda^2\qquad\text{and}\qquad R_{ab}=-4\lambda^2e_a\wedge e_b.$$

Then, the covariant derivative of $L_\omega\psi$ turns into

$$\begin{aligned} \nabla_{X_a}L_\omega\psi &= \lambda\frac{p}{p+1}i_{X_a}d\omega.\psi+\lambda^2p(-e_a\wedge\omega+i_{X_a}\omega).\psi \\ &\quad +2\lambda^2p(e_a\wedge\omega).\psi+\lambda\frac{p}{2(p+1)}(e_a\wedge d\omega-i_{X_a}d\omega).\psi \end{aligned}$$

where we have expanded the Clifford product in terms of wedge and interior products and used the above equality for R_{ab}. Finally, we obtain

$$\begin{aligned} \nabla_{X_a}L_\omega\psi &= \lambda^2p(e_a\wedge\omega+i_{X_a}\omega).\psi+\lambda\frac{p}{2(p+1)}(e_a\wedge d\omega+i_{X_a}d\omega).\psi \\ &= \lambda^2pe_a.\omega.\psi+\lambda\frac{p}{2(p+1)}e_a.d\omega.\psi \\ &= \lambda e_a.L_\omega\psi. \end{aligned}$$

So, we prove that for odd KY forms ω in constant curvature manifolds (or odd special KY forms with $c=\frac{\mathcal{R}}{n(n-1)}$ in general manifolds), the operator

$$L_\omega=(i_{X^a}\omega).\nabla_{X_a}\psi+\frac{p}{2(p+1)}d\omega.\psi$$

is a symmetry operator for Killing spinors.

iii) Twistor spinors
For a conformal Killing vector C with $\mathcal{L}_Cg=2\mu g$, the operator $\pounds_C-\frac{1}{2}\mu$ is a symmetry operator for the twistor equation

$$\nabla_X\left(\pounds_C-\frac{1}{2}\mu\right)\psi=\frac{1}{n}\widetilde{X}.\not{D}\left(\pounds_C-\frac{1}{2}\mu\right)\psi.$$

Note that it is different from the case of massless Dirac equation which was $\pounds_C + \frac{1}{2}(n-1)\mu$.
For higher-degree CKY forms ω, we consider the operator

$$L_\omega\psi = (i_{X^a}\omega).\nabla_{X_a}\psi + \frac{p}{2(p+1)}d\omega.\psi + \frac{p}{2(n-p+1)}\delta\omega.\psi$$

which reduces to $\pounds_C - \frac{1}{2}\mu$ for $p=1$ since $\delta\widetilde{C} = -n\mu$.
We investigate that in which condition the operator L_ω is a symmetry operator for a twistor spinor ψ

$$\nabla_X L_\omega\psi = \frac{1}{n}\widetilde{X}.\not{D}L_\omega\psi.$$

By using the twistor equation, L_ω can be written as

$$L_\omega\psi = -(-1)^p\frac{p}{n}\omega.\not{D}\psi + \frac{p}{2(p+1)}d\omega.\psi + \frac{p}{2(n-p+1)}\delta\omega.\psi.$$

By taking the covariant derivative

$$\begin{aligned}\nabla_{X_a}L_\omega\psi &= -(-1)^p\frac{p}{n}\nabla_{X_a}\omega.\not{D}\psi - (-1)^p\frac{p}{n}\omega.\nabla_{X_a}\not{D}\psi \\ &\quad+\frac{p}{2(p+1)}\nabla_{X_a}d\omega.\psi + \frac{p}{2(p+1)}d\omega.\nabla_{X_a}\psi \\ &\quad+\frac{p}{2(n-p+1)}\nabla_{X_a}\delta\omega.\psi + \frac{p}{2(n-p+1)}\delta\omega.\nabla_{X_a}\psi.\end{aligned}$$

Here, we can use the twistor equation $\nabla_{X_a}\psi = \frac{1}{n}e_a.\not{D}\psi$ and the integrability condition $\nabla_{X_a}\not{D}\psi = \dfrac{n}{2}K_a.\psi$ with $K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)}e_a - P_a\right)$. By also expanding Clifford products in terms of wedge products and interior derivatives, we obtain

$$\begin{aligned}\nabla_{X_a}L_\omega\psi &= \Bigg[-(-1)^p\frac{p}{2n(p+1)}i_{X_a}d\omega - (-1)^p\frac{p}{2n(p+1)}e_a\wedge d\omega \\ &\quad+(-1)^p\frac{p}{2n(n-p+1)}e_a\wedge\delta\omega + (-1)^p\frac{p}{2n(n-p+1)}i_{X_a}\delta\omega\Bigg].\not{D}\psi \\ &\quad+\Bigg[-(-1)^p\frac{p}{2}\omega.K_a + \frac{p}{2(p+1)}\nabla_{X_a}d\omega + \frac{p}{2(n-p+1)}\nabla_{X_a}\delta\omega\Bigg].\psi.\end{aligned}$$

A similar calculation for the right hand side of the twistor equation gives

$$\begin{aligned}\frac{1}{n}e_a.\not{D}L_\omega\psi &= \Bigg[-(-1)^p\frac{p}{2n(p+1)}i_{X_a}d\omega - (-1)^p\frac{p}{2n(p+1)}e_a\wedge d\omega \\ &\quad+(-1)^p\frac{p}{2n(n-p+1)}e_a\wedge\delta\omega + (-1)^p\frac{p}{2n(n-p+1)}i_{X_a}\delta\omega\Bigg].\not{D}\psi \\ &\quad+\Bigg[-(-1)^p\frac{p}{2n}e_a.e^b.\omega.K_b + \frac{p}{2n(p+1)}e_a.e^b.\nabla_{X_b}d\omega \\ &\quad+\frac{p}{2n(n-p+1)}e_a.e^b.\nabla_{X_b}\delta\omega\Bigg].\psi\end{aligned}$$

(details of the calculation can be found in Ü. Ertem, Twistor spinors and extended conformal superalgebras, arXiv:1605.13361).
By comparing the last two equations, one can see that the coefficients of $\not{D}\psi$ are equal to each other.
However, to check the equivalence of coefficients of ψ, we need to use the integrability conditions of CKY forms. They can be calculated as (see Ü. Ertem, Lie algebra of conformal Killing-Yano forms, *Class. Quant. Grav.* 33 (2016) 125033)

$$\nabla_{X_a} d\omega = \frac{p+1}{p(n-p+1)} e_a \wedge d\delta\omega + \frac{p+1}{p} R_{ba} \wedge i_{X^b}\omega$$

$$\begin{aligned} \nabla_{X_a}\delta\omega &= -\frac{n-p+1}{(p+1)(n-p)} i_{X_a}\delta d\omega \\ &\quad + \frac{n-p+1}{n-p}\left(i_{X_b}P_a \wedge i_{X^b}\omega + i_{X_b}R_{ca} \wedge i_{X^c} i_{X^b}\omega \right) \end{aligned}$$

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1} = P_a \wedge i_{X^a}\omega + R_{ab} \wedge i_{X^a} i_{X^b}\omega.$$

In constant curvature manfiolds, these equalities satisfy the equivalence of the coefficients of ψ (see Ü. Ertem, Twistor spinors and extended conformal superalgebras, arXiv:1605.13361).
So, L_ω is a symmetry operator of twistor spinors in constant curvature manifolds.
Moreover, we can also consider <u>normal CKY forms</u> in Einstein manifolds which have the following integrability conditions

$$\nabla_{X_a} d\omega = \frac{p+1}{p(n-p+1)} e_a \wedge d\delta\omega + 2(p+1)K_a \wedge \omega$$

$$\nabla_{X_a}\delta\omega = -\frac{n-p+1}{(p+1)(n-p)} i_{X_a}\delta d\omega - 2(n-p+1) i_{X_b}K_a \wedge i_{X_b}\omega$$

$$\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1} = -2(n-p)K_a \wedge i_{X^a}\omega.$$

In that case also we have the equality $\nabla_{X_a} L_\omega\psi = \frac{1}{n} e_a.\not{D} L_\omega\psi$.
So, L_ω is a symmetry operator in Einstein manifolds with ω are normal CKY forms.
In constant curvature manifolds, all CKY forms are normal CKY forms.

In summary, we have the following diagrams of symmetry operators

$$\begin{tikzcd} \not{D}\psi = m\psi \arrow[d] \\ \pounds_K \arrow[d, shift left, "\text{odd}\, p\, (\text{even}\, p\,,\, L_\omega z)"] \\ L_\omega = i_{X^a}\omega.\nabla_{X_a} + \frac{p}{2(p+1)}d\omega \arrow[u, shift left, bend left, "p=1"] \end{tikzcd}
\qquad
\begin{tikzcd} \nabla_X\psi = \lambda\widetilde{X}.\psi \arrow[d] \\ \pounds_K \arrow[d, shift left, "\text{odd}\, p, \text{constant curvature}"] \\ L_\omega = i_{X^a}\omega.\nabla_{X_a} + \frac{p}{2(p+1)}d\omega \arrow[u, shift left, bend left, "p=1"] \end{tikzcd}$$

and

$$
\begin{tikzcd}
\not{D}\psi = 0 \arrow[d] \\
\pounds_C + \frac{1}{2}(n-1)\mu \arrow[d, shift left] \\
L_\omega = i_{X^a}\omega.\nabla_{X_a} + \frac{p}{2(p+1)}d\omega - \frac{n-p}{2(n-p+1)}\delta\omega \arrow[u, bend left, "p=1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\nabla_X\psi = \frac{1}{n}\widetilde{X}.\not{D}\psi \arrow[d] \\
\pounds_C - \frac{1}{2}\mu \arrow[d, shift left, "\text{const. curv., Einstein}"] \\
L_\omega = i_{X^a}\omega.\nabla_{X_a} + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega \arrow[u, bend left, "p=1"]
\end{tikzcd}
$$

Lecture 6

Extended Superalgebras

6.1 Symmetry superalgebras

A superalgebra $\mathfrak{g}$ is a $\mathbb{Z}_2$-graded algebra which can be written as a direct sum of two components;

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

The even part $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and
the odd part $\mathfrak{g}_1$ is a module of $\mathfrak{g}_0$, i.e., $\mathfrak{g}_0$ acts on $\mathfrak{g}_1$; $\mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$.
The product rule is given by a bilinear operation $[\, ,\,]$;

$$[\, ,\,] : \mathfrak{g}_i \times \mathfrak{g}_j \longrightarrow \mathfrak{g}_{i+j} \quad , \quad i, j = 0, 1$$

so we have

$$[\, ,\,] : \mathfrak{g}_0 \times \mathfrak{g}_0 \longrightarrow \mathfrak{g}_0$$
$$[\, ,\,] : \mathfrak{g}_0 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_1$$
$$[\, ,\,] : \mathfrak{g}_1 \times \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$$

If $[\, ,\,]$ is antisymmetric on $\mathfrak{g}_0$; for $a, b \in \mathfrak{g}_0$ $[a, b] = -[b, a]$
and satisfies the Jacobi identity, for $a, b, c \in \mathfrak{g}_0$ $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$
then $\mathfrak{g}_0$ is a Lie algebra.
Moreover, if $[\, ,\,]$ is symmetric on $\mathfrak{g}_1$, for $a, b \in \mathfrak{g}_1$ $[a, b] = [b, a]$
and $[\, ,\,]$ on $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ satisfies the graded Jacobi identity

$$[a, [b, c]] - [[a, b], c] - (-1)^{|a||b|}[b, [a, c]] = 0$$

for $a, b, c \in \mathfrak{g}$ and $|a|$ denotes the degree of a which corresponds to 0 or 1 depnding on that a is in $\mathfrak{g}_0$ or $\mathfrak{g}_1$, respectively, then $\mathfrak{g}$ is called as a Lie superalgebra.
The dimension of a superalgebra is denoted by $(\alpha|\beta)$ where α is the dimension of $\mathfrak{g}_0$ and β is the dimension of $\mathfrak{g}_1$.

i) Killing-Yano superalgebra

Killing vector fields on a manifold M have a Lie algebra structure w.r.t. the Lie bracket of vector fields;

$$[K_i, K_j] = c_{ijk} K_k$$

which is called the isometry algebra.
For KY forms ω which satisfy $\nabla_X \omega = \frac{1}{p+1} i_X d\omega$, we can define the Schouten-Nijenhuis (SN) bracket $[\,,\,]_{SN}$.
For a p-form α and a q-form β, $[\,,\,]_{SN}$ is defined as follows

$$[\alpha, \beta]_{SN} = i_{X^a}\alpha \wedge \nabla_{X_a}\beta + (-1)^{pq} i_{X^a}\beta \wedge \nabla_{X_a}\alpha$$

and gives a $(p+q-1)$-form.
For $p = q = 1$, $[\,,\,]_{SN}$ reduces to the Lie bracket of vector fields

$$[X, Y] = \nabla_X Y - \nabla_Y X \quad , \quad X, Y \in TM$$

SN bracket satisfies the following graded Lie bracket properties

$$\begin{aligned} [\alpha, \beta]_{SN} &= (-1)^{pq}[\beta, \alpha] \\ (-1)^{p(r+1)}[\alpha, [\beta, \gamma]_{SN}]_{SN} &+ (-1)^{q(p+1)}[\beta, [\gamma, \alpha]_{SN}]_{SN} + (-1)^{r(q+1)}[\gamma, [\alpha, \beta]_{SN}]_{SN} = 0 \end{aligned}$$

where γ is a r-form.

Do KY forms satisfy a Lie algebra under $[\,,\,]_{SN}$?
For a KY p-form ω_1 and KY q-form ω_2; do we have the following equality?

$$\nabla_{X_a}[\omega_1, \omega_2]_{SN} = \frac{1}{p+q} i_{X_a} d[\omega_1, \omega_2]_{SN} \tag{6.1}$$

For LHS, we have

$$\begin{aligned} \nabla_{X_a}[\omega_1, \omega_2]_{SN} &= \nabla_{X_a}\left(i_{X^b}\omega_1 \wedge \nabla_{X_b}\omega_2 + (-1)^{pq} i_{X^b}\omega_2 \wedge \nabla_{X_b}\omega_1\right) \\ &= \nabla_{X_a} i_{X^b}\omega_1 \wedge \nabla_{X_b}\omega_2 + i_{X^b}\omega_1 \wedge \nabla_{X_a}\nabla_{X_b}\omega_2 \\ &\quad + (-1)^{pq}\nabla_{X_a} i_{X^b}\omega_2 \wedge \nabla_{X_b}\omega_1 + (-1)^{pq} i_{X^b}\omega_2 \wedge \nabla_{X_a}\nabla_{X_b}\omega_1 . \end{aligned}$$

Here, we can use $[i_{X^b}, \nabla_{X_a}] = 0$ and the KY equation

$$\begin{aligned} \nabla_{X_a}[\omega_1, \omega_2]_{SN} &= \frac{1}{(p+1)(q+1)}\left(i_{X^b} i_{X_a} d\omega_1 \wedge i_{X_b} d\omega_2 + (-1)^{pq} i_{X^b} i_{X_a} d\omega_2 \wedge i_{X_b} d\omega_1\right) \\ &\quad + \frac{1}{q+1} i_{X^b}\omega_1 \wedge \nabla_{X_a} i_{X_b} d\omega_2 + \frac{(-1)^{pq}}{p+1} i_{X^b}\omega_2 \wedge \nabla_{X_a} i_{X_b} d\omega_1 \end{aligned}$$

again by using $\nabla_{X_a} i_{X_b} = i_{X_b}\nabla_{X_a}$ and from the integrability condition of KY forms; $\nabla_{X_a} d\omega = \frac{p+1}{p} R_{ba} \wedge i_{X^b}\omega$, we find

$$\begin{aligned} \nabla_{X_a}[\omega_1, \omega_2]_{SN} &= \frac{1}{(p+1)(q+1)}\left(i_{X^b} i_{X_a} d\omega_1 \wedge i_{X_b} d\omega_2 + (-1)^{pq} i_{X^b} i_{X_a} d\omega_2 \wedge i_{X_b} d\omega_1\right) \\ &\quad + \frac{1}{q} i_{X^b}\omega_1 \wedge i_{X_b}\left(R_{ca} \wedge i_{X^c}\omega_2\right) + \frac{(-1)^{pq}}{p} i_{X^b}\omega_2 \wedge i_{X_b}\left(R_{ca} \wedge i_{X^c}\omega_1\right) . \end{aligned}$$

For the RHS of (6.1)

$$\frac{1}{p+q}i_{X_a}d[\omega_1,\omega_2]_{SN}=\frac{1}{p+q}i_{X_a}d\left(i_{X^a}\omega_1\wedge\nabla_{X_a}\omega_2+(-1)^{pq}i_{X^a}\omega_2\wedge\nabla_{X_a}\omega_1\right)$$

from the properties $d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^p\alpha\wedge d\beta$, $i_X(\alpha\wedge\beta)=i_X\alpha\wedge\beta+(-1)^p\alpha\wedge i_X\beta$ and $[\nabla_{X_a},d]=0$ in normal coordinates, we have

$$\begin{aligned}\frac{1}{p+q}i_{X_a}d[\omega_1,\omega_2]_{SN} \quad &= \quad \frac{1}{(p+1)(q+1)}\left(i_{X^b}i_{X_a}d\omega_1\wedge i_{X_b}d\omega_2+(-1)^{pq}i_{X^b}i_{X_a}d\omega_2\wedge i_{X_b}d\omega_1\right)\\ &\quad -\frac{(-1)^p}{pq}i_{X_a}\left(i_{X^b}\omega_1\wedge R_{cb}\wedge i_{X^c}\omega_2\right).\end{aligned}$$

In general, we do not have the equality (6.1).
However, for constant curvature manifolds $R_{ab}=ce_a\wedge e_b$, the curvature terms are equal to each other and we have the equality (6.1).
So, KY forms satisfy a Lie superalgebra in constant cuvature manifolds;

$$\mathfrak{k}=\mathfrak{k}_0\oplus\mathfrak{k}_1$$

The even part $\mathfrak{k}_0$ corresponds to odd degree KY forms
The odd part $\mathfrak{k}_1$ corresponds to even degree KY forms.

$$[\,,\,]_{SN}:\mathfrak{k}_0\times\mathfrak{k}_0\longrightarrow\mathfrak{k}_0\quad(\text{if } p \text{ and } q \text{ odd, then } p+q-1 \text{ odd})$$

$$[\,,\,]_{SN}:\mathfrak{k}_0\times\mathfrak{k}_1\longrightarrow\mathfrak{k}_1\quad(\text{if } p \text{ odd and } q \text{ even, then } p+q-1 \text{ even})$$

$$[\,,\,]_{SN}:\mathfrak{k}_1\times\mathfrak{k}_1\longrightarrow\mathfrak{k}_0\quad(\text{if } p \text{ and } q \text{ even, then } p+q-1 \text{ odd})$$

and $[\,,\,]_{SN}$ satisfies the graded Jacobi identity.

From the integrabilty condition of KY forms $\nabla_{X_a}d\omega=\frac{p+1}{p}R_{ba}\wedge i_{X^b}\omega$, the second and higher order derivatives of KY forms can be written in terms of themselves.
By using the integrabilty condition of KY forms, the maximal number of KY p-forms in n-dimensions can be found by counting the independent degrees of freedom of ω and $d\omega$ as

$$K_p=\begin{pmatrix} n\\ p\end{pmatrix}+\begin{pmatrix} n\\ p+1\end{pmatrix}=\begin{pmatrix} n+1\\ p+1\end{pmatrix}=\frac{(n+1)!}{(p+1)!(n-p)!}$$

and this number is achived in constant curvature manifolds.
So, the diemsnion of the KY superalgebra is $(K_{odd}|K_{even})$ where

$$K_{odd}=\sum_{k=1}^{\lfloor\frac{n}{2}\rfloor}\begin{pmatrix} n+1\\ 2k\end{pmatrix}\quad\text{and}\quad K_{even}=\sum_{k=1}^{\lfloor\frac{n-1}{2}\rfloor}\begin{pmatrix} n+1\\ 2k+1\end{pmatrix}$$

ii) Conformal Killing-Yano superalgebra

Conformal Killing vector fields also satisfy a Lie algebra w.r.t. the Lie bracket of vector fields

$$[C_i, C_j] = f_{ijk} C_k$$

which is called the conformal algebra.
For CKY forms ω; $\nabla_X \omega = \frac{1}{p+1} i_X d\omega - \widetilde{X} \wedge \delta\omega$, we can define the CKY bracket $[\,,\,]_{CKY}$.
For a CKY p-form ω_1 and a CKY q-form ω_2, $[\,,\,]_{CKY}$ is defined as

$$\begin{aligned} [\omega_1, \omega_2]_{CKY} &= \frac{1}{q+1} i_{X^a}\omega_1 \wedge i_{X_a} d\omega_2 + \frac{(-1)^p}{p+1} i_{X^a} d\omega_1 \wedge i_{X_a}\omega_2 \\ &\quad + \frac{(-1)^p}{n-q+1}\omega_1 \wedge \delta\omega_2 + \frac{1}{n-p+1}\delta\omega_1 \wedge \omega_2. \end{aligned}$$

Note that it is different from the SN bracket, but a slight modification of it.
$[\,,\,]_{CKY}$ satisfies the graded Lie bracket properties.
For a CKY p-form ω_1 and a CKY q-form ω_2, $[\omega_1, \omega_2]_{CKY}$ is a CKY $(p+q-1)$-form, i.e.

$$\nabla_{X_a}[\omega_1, \omega_2]_{CKY} = \frac{1}{p+q} i_{X_a} d[\omega_1, \omega_2]_{CKY} - \frac{1}{n-p-q+2} e_a \wedge \delta[\omega_1, \omega_2]_{CKY}$$

in constant curvature manifolds and in Einstein manifolds for normal CKY forms (see Ü. Ertem, Lie algebra of conformal Killing-Yano forms, *Class. Quant. Grav.* 33 (2016) 125033).
So, CKY forms satisfy a Lie superalgebra in these cases,

$$\mathfrak{c} = \mathfrak{c}_0 \oplus \mathfrak{c}_1$$

The even part $\mathfrak{c}_0$ corresponds to odd degree CKY forms
The odd part $\mathfrak{c}_1$ corresponds to even degree CKY forms

$$[\,,\,]_{CKY} : \mathfrak{c}_0 \times \mathfrak{c}_0 \longrightarrow \mathfrak{c}_0$$

$$[\,,\,]_{CKY} : \mathfrak{c}_0 \times \mathfrak{c}_1 \longrightarrow \mathfrak{c}_1$$

$$[\,,\,]_{CKY} : \mathfrak{c}_1 \times \mathfrak{c}_1 \longrightarrow \mathfrak{c}_0$$

From the integrability condition of CKY forms $\frac{p}{p+1}\delta d\omega + \frac{n-p}{n-p+1} d\delta\omega = P_a \wedge i_{X^a}\omega + R_{ab} \wedge i_{X^a} i_{X^b}\omega$, the counting of the independent degrees of freedom of ω, $d\omega$, $\delta\omega$ and $d\delta\omega$ gives the maximal number of CKY p-forms in n-dimensions (which is achieved in constant curvature manifolds)

$$C_p = 2\begin{pmatrix} n \\ p \end{pmatrix} + \begin{pmatrix} n \\ p-1 \end{pmatrix} + \begin{pmatrix} n \\ p+1 \end{pmatrix} = \begin{pmatrix} n+2 \\ p+1 \end{pmatrix} = \frac{(n+2)!}{(p+1)!(n-p+1)!}$$

So, the dimension of the CKY superalgebra is $(C_{odd}|C_{even})$ where

$$C_{odd} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} n+2 \\ 2k \end{pmatrix} \quad \text{and} \quad C_{even} = \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \begin{pmatrix} n+2 \\ 2k+1 \end{pmatrix}$$

All KY forms are CKY forms at the same time and the number of CKY p-forms that do not correspond to KY p-forms is given by

$$C_p - K_p = \left(\begin{array}{c} n+2 \\ p+1 \end{array} \right) - \left(\begin{array}{c} n+1 \\ p+1 \end{array} \right) = \frac{(n+1)!}{p!(n-p+1)!}$$

6.2 Extended superalgebras

i) Killing superalgebras
We can define a superalgebra structure by using Killing vectors and Killing spinors

$$\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$$

The even part $\mathfrak{k}_0$ corresponds to Lie algebra of Killing vector fields
The odd part $\mathfrak{k}_1$ corresponds to the set of Killing spinors
The brackets of the superalgebra is defined as follows;
The even-even bracket is the Lie bracket of Killing vector fields

$$\begin{array}{rcl} [\,,\,] : \mathfrak{k}_0 \times \mathfrak{k}_0 & \longrightarrow & \mathfrak{k}_0 \\ (K_1, K_2) & \longmapsto & [K_1, K_2] \end{array}$$

The even-odd bracket is the Lie derivative on spinor fields

$$\begin{array}{rcl} \pounds : \mathfrak{k}_0 \times \mathfrak{k}_1 & \longrightarrow & \mathfrak{k}_1 \\ (K, \psi) & \longmapsto & \pounds_K \psi \end{array}$$

The odd-odd bracket is the Dirac currents of Killling spinors

$$\begin{array}{rcl} (\,)_1 : \mathfrak{k}_1 \times \mathfrak{k}_1 & \longrightarrow & \mathfrak{k}_0 \\ (\psi, \phi) & \longmapsto & (\psi\overline{\phi})_1 \cong V_{\psi,\phi} \end{array}$$

The Jacobi identities correspond to

$$\begin{array}{l} [K_1, [K_2, K_3]] + [K_2, [K_3, K_1]] + [K_3, [K_1, K_2]] = 0 \\ [\pounds_{K_1}, \pounds_{K_2}]\psi = \pounds_{[K_1,K_2]}\psi \\ \mathcal{L}_K(\psi\overline{\phi}) = (\pounds_K\psi)\overline{\phi} + \psi(\overline{\pounds_K\phi}) \\ \pounds_{V_\psi}\psi = 0 \end{array}$$

The first three identities are satisfied automatically from the properties of the Lie derivative, but the last one is not satisfied automatically.
For manifolds on which the last identity is satisfied, Killing superalgebra is a Lie superalgebra.
We can extend this superalgebra structure to include higher-degree KY forms in constant curvature manifolds,

$$\overline{\mathfrak{k}} = \overline{\mathfrak{k}}_0 \oplus \overline{\mathfrak{k}}_1$$

The even part $\bar{\mathfrak{k}}_0$ corresponds to the Lie algebra of odd KY forms
The odd part $\bar{\mathfrak{k}}_1$ corresponds to the set of Killing spinors
The brackets of the extended superalgebra are defined as follows;
The even-even bracket is the SN bracket of KY forms

$$\begin{aligned} [\,,\,]_{SN} : \bar{\mathfrak{k}}_0 \times \bar{\mathfrak{k}}_0 &\longrightarrow \bar{\mathfrak{k}}_0 \\ (\omega_1, \omega_2) &\longmapsto [\omega_1, \omega_2]_{SN} \end{aligned}$$

The even-odd bracket is the symmetry operators of Killing spinors

$$\begin{aligned} L : \bar{\mathfrak{k}}_0 \times \bar{\mathfrak{k}}_1 &\longrightarrow \bar{\mathfrak{k}}_1 \\ (\omega, \psi) &\longmapsto L_\omega \psi = (i_{X^a}\omega).\nabla_{X_a}\psi + \frac{p}{2(p+1)} d\omega.\psi \end{aligned}$$

The odd-odd bracket is the p-form Dirac currents of Killling spinors

$$\begin{aligned} (\,)_p : \bar{\mathfrak{k}}_1 \times \bar{\mathfrak{k}}_1 &\longrightarrow \bar{\mathfrak{k}}_0 \\ (\psi, \phi) &\longmapsto (\psi\bar{\phi})_p \end{aligned}$$

The Jacobi identities are

$$\begin{aligned} &[\omega_1, [\omega_2, \omega_3]_{SN}]_{SN} + [\omega_2, [\omega_3, \omega_1]_{SN}]_{SN} + [\omega_3, [\omega_1, \omega_2]_{SN}]_{SN} = 0 \\ &[L_{\omega_1}, L_{\omega_2}]\psi = L_{[\omega_1, \omega_2]_{SN}}\psi \\ &[\omega, (\psi\bar{\phi})]_{SN} = (L_\omega\psi)\bar{\phi} + \psi(\overline{L_\omega\phi}) \\ &L_{(\psi\bar{\psi})_p}\psi = 0 \end{aligned}$$

The first identity is satisfied from the properties of the SN bracket, but the last three identities are not satisfied.
So, $\bar{\mathfrak{k}}$ is a superalgebra in constant curvature manifolds, but not a Lie superalgebra.
For $n \leq 5$, we can also define a new bracket for KY forms (p-form ω_1 and q-form ω_2),

$$[\omega_1, \omega_2]_{KY} = \frac{pq}{p+q-1}[\omega_1, \omega_2]_{SN} - \frac{pq}{p+q}[i_{X^a} i_{x^b}\omega_1, i_{X_a} i_{X_b}\omega_2]_{SN}$$

which is a Lie bracket and satisfies the second Jacobi identity

$$[L_{\omega_1}, L_{\omega_2}]\psi = L_{[\omega_1, \omega_2]_{KY}}\psi$$

(see Ü. Ertem, Symmetry operators of Killing spinors and superalgebras in AdS_5, *J. Math. Phys.* 2016)

ii) Conformal superalgebras
We can define a superalgebra structure by using conformal Killing vectors and twistor spinors,

$$\mathfrak{c} = \mathfrak{c}_0 \oplus \mathfrak{c}_1$$

The even part $\mathfrak{c}_0$ corresponds to the Lie algebra of conformal Killing vectors
The odd part $\mathfrak{c}_1$ corresponds to the set of twistor spinors
The brackets of the superalgebra are as follows;
The even-even bracket is the Lie bracket of conformal Killing vector fields

$$\begin{aligned} [\,,\,] : \mathfrak{c}_0 \times \mathfrak{c}_0 &\longrightarrow \mathfrak{c}_0 \\ (C_1, C_2) &\longmapsto [C_1, C_2] \end{aligned}$$

The even-odd bracket is the Lie derivative on twistor spinors

$$\begin{aligned} \pounds - \frac{1}{2}\mu : \mathfrak{c}_0 \times \mathfrak{c}_1 &\longrightarrow \mathfrak{c}_1 \\ (C, \psi) &\longmapsto \pounds_C\psi - \frac{1}{2}\mu\psi \end{aligned}$$

The odd-odd bracket is the Dirac currents of twistor spinors

$$\begin{aligned} (\,)_1 : \mathfrak{c}_1 \times \mathfrak{c}_1 &\longrightarrow \mathfrak{c}_0 \\ (\psi, \phi) &\longmapsto (\psi\overline{\phi})_1 \cong V_{\psi,\phi} \end{aligned}$$

The Jacobi identities are not satisfied automatically.
We can extend this superalgebra structure to include higher-degree CKY forms in constant curvature manifolds or Einstein manifolds with normal CKY forms

$$\bar{\mathfrak{c}} = \bar{\mathfrak{c}}_0 \oplus \bar{\mathfrak{c}}_1$$

The even part $\bar{\mathfrak{c}}_0$ corresponds to the Lie algebra of CKY forms (or normal CKY forms)
The even part $\bar{\mathfrak{c}}_1$ corresponds to the set of twistor spinors
The brackets of the extended conformal superalgebra are defined as The even-even bracket is the CKY bracket of CKY forms

$$\begin{aligned} [\,,\,]_{CKY} : \bar{\mathfrak{c}}_0 \times \bar{\mathfrak{c}}_0 &\longrightarrow \bar{\mathfrak{c}}_0 \\ (\omega_1, \omega_2) &\longmapsto [\omega_1, \omega_2]_{CKY} \end{aligned}$$

The even-odd bracket is the symmetry operators of twistor spinors

$$\begin{aligned} L : \bar{\mathfrak{c}}_0 \times \bar{\mathfrak{c}}_1 &\longrightarrow \bar{\mathfrak{c}}_1 \\ (\omega, \psi) &\longmapsto L_\omega\psi = (i_{X^a}\omega).\nabla_{X_a}\psi + \frac{p}{2(p+1)}d\omega.\psi + \frac{p}{2(n-p+1)}\delta\omega.\psi \end{aligned}$$

The odd-odd bracket is the p-form Dirac currents of twistor spinors

$$\begin{aligned} (\,)_p : \bar{\mathfrak{c}}_1 \times \bar{\mathfrak{c}}_1 &\longrightarrow \bar{\mathfrak{c}}_0 \\ (\psi, \phi) &\longmapsto (\psi\overline{\phi})_p \end{aligned}$$

The Jacobi identities are not satisfied automatically.
So, $\bar{\mathfrak{c}}$ is a superalgebra in constant curvature manifolds (or in Einstein manifolds with normal CKY forms), but not a Lie superalgebra.

Lecture 7

Harmonic Spinors from Twistors

7.1 Twistors to harmonic spinors

Let us start with a twistor spinor ψ which satisfies $\nabla_X \psi = \frac{1}{n}\widetilde{X}.\not{D}\psi$, and consider a function f which is a solution of the conformally generalized Laplace equation in n-dimensions;

$$\Delta f - \frac{n-2}{4(n-1)}\mathcal{R}f = 0$$

where $\Delta = -d\delta - \delta d$ is the Laplace-Beltrami operator and $\mathcal{R}$ is the curvature scalar.
We can define an operator

$$L_f = \frac{n-2}{n} f \not{D} + df$$

and the action of this operator on a twistor spinor gives a harmonic spinor, i.e., we have $\not{D} L_f \psi = 0$. This can be seen as follows

$$\begin{aligned}
\not{D} L_f \psi &= e^a.\nabla_{X_a}\left(\frac{n-2}{n} f \not{D}\psi + df.\psi\right) \\
&= e^a.\left(\frac{n-2}{n}(\nabla_{X_A} f)\not{D}\psi + \frac{n-2}{n} f \nabla_{X_a}\not{D}\psi + \nabla_{X_a} df.\psi + df.\nabla_{X_a}\psi\right) \\
&= \frac{n-2}{n}\not{d}f.\not{D}\psi + \frac{n-2}{n} f \not{D}^2\psi + \not{d}df.\psi + e^a.df.\nabla_{X_a}\psi
\end{aligned}$$

where we have used $e^a.\nabla_{X_a} f = (d-\delta)f = \not{d}f$, $e^a.\nabla_{X_a}\not{D}\psi = \not{D}^2\psi$ and $e^a.\nabla_{X_a} df = \not{d}df$.
Since f is a function; $\delta f = 0$, $\not{d}f = df$ and $\not{d}df = -\delta df = \Delta f$ (since $d^2 = 0$).
From the twistor equation and the integrability condition of twistor spinors

$\not{D}^2\psi = -\dfrac{n}{4(n-1)}\mathcal{R}\psi$, we can write

$$\not{D}L_f\psi = \frac{n-2}{n}\left(df.\not{D}\psi - \frac{n}{4(n-1)}\mathcal{R}f\psi\right) + \Delta f.\psi + \frac{1}{n}e^a.df.e_a.\not{D}\psi.$$

We know that f satisfies generalized Laplace equation and we have the property $e^a.\alpha.e_a = (-1)^p(n-2p)\alpha$ for a p-form α. So, we have

$$\not{D}L_f\psi = \left(\Delta f - \frac{n-2}{4(n-1)}\mathcal{R}f\right).\psi = 0.$$

Can we generalize the operator L_f to higher-degree differential forms?
We propose the following operator for a p-form α and an inhomogeneous Clifford form Ω;

$$L_\alpha = \alpha.\not{D} + \Omega.$$

In which conditions we can have a harmonic spinor by applying L_α to a twistor spinor ψ, i.e., $\not{D}L_\alpha\psi = 0$?
We can calculate explicitly as

$$\begin{aligned}\not{D}L_\alpha\psi &= e^a.\nabla_{X_a}L_\alpha\psi \\ &= e^a.\nabla_{X_a}\alpha.\not{D}\psi + e^a.\alpha.\nabla_{X_a}\not{D}\psi + e^a.\nabla_{X_a}\Omega.\psi + e^a.\Omega.\nabla_{X_a}\psi.\end{aligned}$$

From the definition $\not{d} = e^a.\nabla_{X_a}$, twistor equation and the integrability condition $\nabla_{X_a}\not{D}\psi = \dfrac{n}{2}K_a.\psi$, we can write

$$\begin{aligned}\not{D}L_\alpha\psi &= \not{d}\alpha.\not{D}\psi + \frac{n}{2}e^a.\alpha.K_a.\psi + \not{d}\Omega.\psi + \frac{1}{n}e^a.\Omega.e_a.\not{D}\psi \\ &= \left(\not{d}\alpha + \frac{n-2\Pi}{n}\eta\Omega\right).\not{D}\psi + \left(\not{d}\Omega + \frac{n}{2}e^a.\alpha.K_a\right).\psi.\end{aligned}$$

Here we have used $e^a.\Omega.e_a = (n-2\Pi)\eta\Omega$ where $\Pi\Omega = e^a \wedge i_{X_a}\Omega$.
Hence, if $L_\alpha\psi$ would be a harmonic spinor, then we have two conditions

$$\not{d}\alpha + \frac{n-2\Pi}{n}\eta\Omega = 0 \tag{7.1}$$

$$\not{d}\Omega + \frac{n}{2}e^a.\alpha.K_a = 0. \tag{7.2}$$

As a special case, in even dimensions $n = 2k$, we can choose Ω as a k-form (middle form) and (7.1) gives $d\alpha = -\delta\alpha$ and this can be possible for $\alpha = 0$ since α is a homogeneous p-form. In that case, (7.2) gives that Ω is a harmonic form; $d\Omega = \delta\Omega = 0$.
So, in even dimensions, for a harmonic middle form Ω and a twistor spinor ψ, $\Omega.\psi$ is a harmonic spinor.

In general, if we choose α a non-zero p-form, then (7.1) gives that Ω is a sum of $(p+1)$ and $(p-1)$-forms. So, we have from (7.1)

$$\Omega = \frac{(-1)^p n}{n-2(p+1)}d\alpha - \frac{(-1)^p n}{n-2(p-1)}\delta\alpha.$$

By applying the Hodge-de Rham operator $\not{d} = d - \delta$

$$\begin{aligned} \not{d}\Omega &= \frac{(-1)^p n}{n-2(p+1)}\not{d}d\alpha - \frac{(-1)^p n}{n-2(p-1)}\not{d}\delta\alpha \\ &= -\frac{(-1)^p n}{n-2(p+1)}\delta d\alpha - \frac{(-1)^p n}{n-2(p-1)}d\delta\alpha \end{aligned}$$

and by substituting in (7.2), we have

$$\frac{(-1)^p}{n-2(p+1)}\delta d\alpha + \frac{(-1)^p}{n-2(p-1)}d\delta\alpha = \frac{1}{2}e^a.\alpha.K_a.$$

From the definiton $K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)}e_a - P_a\right)$, the RHS can be written explicitly as

$$e^a.\alpha.K_a = (-1)^p\frac{2}{n-2}P_a \wedge i_{X^a}\alpha - (-1)^p\frac{n+2(p-1)}{2(n-1)(n-2)}\mathcal{R}\alpha.$$

So, we obtain

$$\frac{1}{n-2(p+1)}\delta d\alpha + \frac{1}{n-2(p-1)}d\delta\alpha = \frac{1}{n-2}P_a \wedge i_{X^a}\alpha - \frac{n+2(p-1)}{4(n-1)(n-2)}\mathcal{R}\alpha.$$

A p-form α that satisfies this equation is called as a potential form. Because, in even dimensions, for $p = \frac{n}{2} - 1$, α is a potential form for middle-form Maxwell equations and for $p = \frac{n}{2} + 1$ it is a co-potential form for the same equations. For $p = 0$, potential form equation reduces to the conformally generalized Laplace equation. So, it is generalization of the conformal Laplace equation. Then, we obtain that for a potential p-form α and a twistor spinor ψ

$$L_\alpha\psi = \alpha.\not{D}\psi + \frac{(-1)^p n}{n-2(p+1)}d\alpha.\psi - \frac{(-1)^p n}{n-2(p-1)}\delta\alpha.\psi$$

is a harmonic spinor.

Remember the symmetry operators of twistor spinors which are defined in terms of CKY forms in constant curvature manifolds and normal CKY forms in Einstein manifolds;

$$L_\omega = -(-1)^p\frac{p}{n}\omega.\not{D} + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega$$

and the symmetry operators of harmonic spinors (massless Dirac equation) in terms of CKY forms

$$\mathcal{L}_\omega = (i_{x^a}\omega).\nabla_{X_a} + \frac{p}{2(p+1)}d\omega - \frac{n-p}{2(n-p+1)}\delta\omega.$$

So, we have the following relations

$$\text{twistor spinors} \xrightarrow[\text{CKY forms}]{L_\omega} \text{twistor spinors}$$

$$\text{twistor spinors} \xrightarrow[\text{potential forms}]{L_\alpha} \text{harmonic spinors}$$

$$\text{harmonic spinors} \xrightarrow[\text{CKY forms}]{\mathcal{L}_\omega} \text{harmonic spinors}$$

7.2 Gauged twistors to gauged harmonic spinors

We consider a manifold M with a $Spin^c$ structure which means that we can lift the complexified bundle $SO(n) \times_{\mathbb{Z}_2} U(1)$ to $Spin^c(n) = Spin(n) \times_{\mathbb{Z}_2} U(1) = Spin(n) \times U(1)/\mathbb{Z}_2$.
Note that $Spin^c(n) \subset Cl(n) \otimes \mathbb{C} = Cl^c(n)$ is a subgroup of the complexified Clifford algebra.
In that case, we have the $Spin^c$ spinor bundle $\Sigma^c M = \Sigma M \times_{\mathbb{Z}_2} U(1)$ and the connection on $\Sigma^c M$ consists of two parts; the Levi-Civita part on ΣM and the $U(1)$-connection part.
The gauged connection $\widehat{\nabla}$ on $\Sigma^c M$ is written as follows

$$\widehat{\nabla}_X = \nabla_X + i_X A \quad , \quad X \in TM.$$

Gauged exterior and co-derivatives are defined as

$$\widehat{d} = e^a \wedge \widehat{\nabla}_{X_a} = d + A\wedge$$

$$\widehat{\delta} = -i_{X^a}\widehat{\nabla}_{X_a} = \delta - i_{\widetilde{A}}$$

where $\widetilde{A}$ is the vector field corresponding to the metric dual of A.
However, unlike the case of $d^2 = 0 = \delta^2$, we have

$$\widehat{d}^2 = F\wedge$$

$$\widehat{\delta}^2 = -(i_{X^a} i_{X^b} F) i_{X_a} i_{X_b}$$

where $F = dA$ is the gauge curvature.
The gauged Hodge-de Rham operator is

$$\widehat{\displaystyle{\not}d} = \widehat{d} - \widehat{\delta} = {\not}d + A.$$

Gauged curvature operator is

$$\begin{aligned} \widehat{R}(X_a, X_b) &= [\widehat{\nabla}_{X_a}, \widehat{\nabla}_{X_b}] - \widehat{\nabla}_{[X_a, X_b]} \\ &= R(X_a, X_b) - i_{X_a} i_{X_b} F \end{aligned}$$

and the gauged Dirac operator acting on $\Sigma^c M$ is written as

$$\widehat{{\not}D} = e^a.\widehat{\nabla}_{X_a} = {\not}D + A$$

and its square is equal to $\widehat{\not{D}}^2 = \widehat{\nabla}^2 - \frac{1}{4}\mathcal{R} + F$.
We can define the gauged twistor equation

$$\widehat{\nabla}_{X_a}\psi = \frac{1}{n}e_a.\widehat{\not{D}}\psi$$

and the gauged harmonic spinor equation

$$\widehat{\not{D}}\psi = 0.$$

The integrability conditions of gauged twistor spinors can be obtained as

$$\begin{aligned}
\widehat{\not{D}}^2\psi &= -\frac{n}{4(n-1)}\mathcal{R}\psi + \frac{n}{n-1}F.\psi \\
\widehat{\nabla}_{X_a}\widehat{\not{D}}\psi &= \frac{n}{2}K_a.\psi - \frac{n}{(n-1)(n-2)}e_a.F.\psi + \frac{n}{n-2}i_{X_a}F.\psi \\
C_{ab}.\psi &= 2(i_{X_a}i_{X_b}F)\psi + \frac{n}{n-2}\left(e_b.i_{X_a}F - e_a.i_{X_b}F\right).\psi \\
&\quad + \frac{4}{(n-1)(n-2)}e_a.e_b.F.\psi.
\end{aligned}$$

In constant curvature manifolds, we have $C_{ab} = 0$ and the existence of gauged twistors implies $F = 0$.
So, we can have gauged twistor spinors w.r.t. flat connections ($A \neq 0$ and $F = 0$).
Symmetry operators of gauged twistor spinors can be constructed from CKY forms in constant curvature manifolds. For a CKY p-form ω

$$\widehat{L}_\omega = -(-1)^p\frac{p}{n}\omega.\widehat{\not{D}} + \frac{p}{2(p+1)}d\omega + \frac{p}{2(n-p+1)}\delta\omega$$

is a symmetry operator, i.e., we have $\widehat{\nabla}_{X_a}\widehat{L}_\omega\psi = \frac{1}{n}e_a.\widehat{\not{D}}\widehat{L}_\omega\psi$ for a gauged twistor spinor ψ.
(see Ü. Ertem, Gauged twistor spinors and symmetry operators, *J. Math. Phys.* 58 (2017) 032302)

Symmetry operators of gauged harmonic spinors can be constructed from gauged CKY forms which satisfy

$$\widehat{\nabla}_X\omega = \frac{1}{p+1}i_X\widehat{d}\omega - \frac{1}{n-p+1}\widetilde{X}\wedge\widehat{\delta}\omega$$

for any $X \in TM$. Symmetry operators are written in terms of a gauged CKY p-form as

$$\widehat{\mathcal{L}}_\omega = (i_{X^a}\omega).\widehat{\nabla}_{X_a} + \frac{p}{2(p+1)}\widehat{d}\omega - \frac{n-p}{2(n-p+1)}\widehat{\delta}\omega$$

i.e, for a harmonic spinor ψ, we have $\widehat{\not{D}}\widehat{\mathcal{L}}_\omega\psi = 0$.

We can also construct transformation operators between gauged twistor spinors and gauged harmonic spinors.
For a function f that satisfies the generalized gauged Laplace equation

$$\widehat{\Delta}f + \left[\left(1 + \frac{n-2}{n-1}\right)\gamma - \frac{n-2}{4(n-1)}\mathcal{R}\right]f = 0$$

we can construct the operator (where γ is a real number)

$$\widehat{L}_f = \frac{n-2}{n} f \widehat{\not{D}} + \widehat{d}f$$

which transforms a gauged twistor spinor ψ with the property $F.\psi = \gamma\psi$ to a gauged harmonic spinor, i.e., $\widehat{\not{D}}\widehat{L}_f\psi = 0$.
For a gauged potential form α which satisfies

$$\frac{1}{n-2(p+1)}\widehat{\delta}\widehat{d}\alpha + \frac{1}{n-2(p-1)}\widehat{d}\widehat{\delta}\alpha = \frac{1}{n-2}P_A \wedge i_{X^a}\alpha - \frac{n+2(p-1)}{4(n-1)(n-2)}\mathcal{R}\alpha$$

the operator

$$\widehat{L}_\alpha = \alpha.\widehat{\not{D}} + \frac{(-1)^p n}{n-2(p+1)}\widehat{d}\alpha - \frac{(-1)^p n}{n-2(p-1)}\widehat{\delta}\alpha$$

transforms a gauged twistor spinor ψ to a gauged harmonic spinor in constant curvature manifolds, i.e., $\widehat{\not{D}}\widehat{L}_\alpha\psi = 0$.
(see Ü. Ertem, Harmonic spinors from twistors and potental forms, arXiv:1704.04741)
So, we have the following diagram in constant curvature manifolds

$$\text{gauged twistor spinors} \xrightarrow[\text{CKY forms}]{\widehat{L}_\omega} \text{gauged twistor spinors}$$

$$\text{gauged twistor spinors} \xrightarrow[\text{gauged potential forms}]{\widehat{L}_\alpha} \text{gauged harmonic spinors}$$

$$\text{gauged harmonic spinors} \xrightarrow[\text{gauged CKY forms}]{\widehat{\mathcal{L}}_\omega} \text{gauged harmonic spinors}$$

7.3 Seiberg-Witten solutions

Let us consider a 4-dimensional manifold M.
Every compact orientable 4-manifold has a $Spin^c$ structure.
In four dimensions, we have $\Sigma^c M = \Sigma^+ M \oplus \Sigma^- M$.
For a spinor $\psi \in \Sigma^+ M$, Seiberg-Witten (SW) equations are written as

$$\begin{aligned} \widehat{\not{D}}\psi &= 0 \\ (\psi\overline{\psi})_2 &= F^+ \end{aligned}$$

where $F^+ = \frac{1}{2}(F + *F)$ is the self-dual curvature 2-form.
So, the solutions of the SW equations are gauged harmonic spinors with a condition on 2-form Dirac currents.

We obtain gauged harmonic spinors from gauged twistor spinors in constant curvature manifolds.
In that case, we have $F = 0$ (with $A \neq 0$), and the second SW equation turns into

$$(\psi\overline{\psi})_2 = 0.$$

This is not a very restrictive condition since we can automatically satisfy this condition by choosing symmetric spinor inner product, i.e., $(\psi, \phi) = (\phi, \psi)$ for $\psi, \phi \in \Sigma^c M$

$$\begin{aligned}(\psi\overline{\psi})_2 &= (\psi, e_b.e_a.\psi)e^a \wedge e^b \\ &= ((e_b.e_a.)^{\mathcal{J}}.\psi, \psi)e^a \wedge e^b \\ &= -(e_b.e_a.\psi, \psi)e^a \wedge e^b \\ &= -(\psi, e_b.e_a.\psi)e^a \wedge e^b \\ &= 0\end{aligned}$$

where we have used that the choice of involution $\mathcal{J} = \xi$ or $\xi\eta$ gives the same sign $(e_b.e_a.)^{\mathcal{J}} = -e_b.e_a.$.
By starting with a gauged twistor spinor ψ, we can construct gauged harmonic spinors from gauged potential, gauged CKY and CKY forms (α, $\widehat{\omega}$ and ω)

$$\widehat{L}_\alpha\psi \quad , \quad \widehat{L}_\omega\widehat{L}_\alpha\psi \quad , \quad \widehat{L}_\alpha\widehat{\mathcal{L}}_{\widehat{\omega}}\psi \quad , \quad \widehat{L}_\omega\widehat{L}_\alpha\widehat{\mathcal{L}}_{\widehat{\omega}}\psi.$$

If they have vanishing 2-form Dirac currents (which is automatically true for symmetric spinor inner product), then we obtain SW solutions w.r.t. flat connections in constant curvature manifolds.
So, we have the relation

$$\text{gauged twistor spinors} \xrightarrow[\text{gauged potential forms}]{\widehat{L}_\alpha} \text{gauged harmonic spinors}$$

$$\text{gauged harmonic spinors} \xrightarrow[\text{vanishing 2-form Dirac current}]{\text{flat connection}} \text{SW solutions}$$

Lecture 8

Supergravity Killing Spinors

8.1 Bosonic supergravity

Supergravity theories are supersymmetric generalizations of General Relativity in various dimensions.
A supergravity action S consists of bosonic (ϕ_i) amd fermionic (ψ_i) fields;

$$S = \text{bosonic } (\phi_i, \nabla\phi_i, ...) + \text{fermionic } (\psi_i, \nabla\psi_i, ...)$$

Bosonic part corresponds to gravitational and gauge field degrees of freedom (graviton (metric), p-form fields, etc) and fermionic part consists of matter degrees of freedom (gravitino (spin-$\frac{3}{2}$ Rarita-Schwinger field), gaugino, etc).
Supersymmetry transformations relate the bosonic and fermionic fields to each other.
For a spinor parameter ϵ, supersymmetry transformations are in the following form

$$\begin{aligned} \delta_\epsilon\phi &= \bar{\epsilon}.\psi && (8.1) \\ \delta_\epsilon\psi &= (\nabla + f(\phi))\,\epsilon && (8.2) \end{aligned}$$

where δ_ϵ denotes the variation of the field.
When we take the fermionic fields to be zero, we obtain bosonic supergravity whose solutions give the consistent backgrounds of the theory.
When $\psi_i \to 0$ we have $\delta_\epsilon\phi_i = 0$ automatically, and $\delta_\epsilon\psi = 0$ gives a constraint on the spinor parameter ϵ that is $(\nabla + f(\phi))\,\epsilon = 0$ which is called the supergravity Killing spinor equation.
So, to obtain a consistent supergravity background, besides the field equations of the theory, we must also solve the supergravity Killing spinor equation.
There are various supergravity theories between the dimensions 4 and 11;

D (dimension)	N (number of susy generators)
4	$1, 2, 3, 4, 5, 6, 8$
5	$2, 4, 6, 8$
6	$(1, 0), (2, 0), (1, 1), (3, 0), (2, 1), (4, 0), (3, 1), (2, 2)$
7	$2, 4$
8	$1, 2$
9	$1, 2$
10	$1(\text{type I} + \text{heterotic}), 2(\text{type IIA} + \text{IIB})$
11	$1(\text{M})$

10 and 11 dimensional supergravity theories correspond to low energy limits of string and M-theories, respectively.

$D = 6$, $N = (1, 0)$ bosonic supergravity action is

$$S = \kappa \int \left(R_{ab} \wedge *(e^a \wedge e^b) - \frac{1}{2} H \wedge *H \right)$$

where H is a 3-form field.
The field equations give the Einsteni field equations and the conditions $dH = 0$ and H is anti-self-dual.
Variation of the gravitino field (Rarita-Schwinger) in the fermionic action gives the following supergravity Killing spinor equation

$$\nabla_X \epsilon + \frac{1}{4} i_X H.\epsilon = 0.$$

$D = 10$, type I and heterotic theories also give the same spinor equation. However, in those cases one also have the extra algebraic constraints due to the existence of extra gaugino fields;

$$\begin{aligned} \left(d\phi + \frac{1}{2} H \right) .\epsilon &= 0 \\ F_2.\epsilon &= 0 \end{aligned}$$

where ϕ is the scalar dilaton and F_2 is a 2-form gauge field.

$D = 11$ bosonic supergravity action is

$$S = \kappa \int \left(R_{ab} \wedge *(e^a \wedge e^b) - \frac{1}{2} F \wedge *F - \frac{1}{6} A \wedge F \wedge F \right)$$

where $F = dA$ is a 4-form gauge field.
Nothe that the first term is the gravity action, the second term is the Maxwell-like term and the third term is the Chern-Simons term.

The field equations of the theory are given by

$$\begin{aligned} Ric(X,Y) * 1 &= \frac{1}{2} i_X F \wedge * i_Y F - \frac{1}{6} g(X,Y) F \wedge * F \\ d * F &= \frac{1}{2} F \wedge F \end{aligned}$$

where $X, Y \in TM$ and Ric is the Ricci tensor.
Variation of the gravitino field gives the following supergravity Killing spinor equation

$$\nabla_X \epsilon + \frac{1}{6} i_X F.\epsilon - \frac{1}{12}(\widetilde{X} \wedge F).\epsilon = 0$$

which can also be written in the form

$$\nabla_X \epsilon + \frac{1}{24}\left(\widetilde{X}.F - 3F.\widetilde{X}\right).\epsilon = 0.$$

$D = 5$ supergravity has a similar structure with the coefficient $\frac{1}{24}$ is replaced by the coefficient $\frac{1}{4\sqrt{3}}$ and F is a 2-form.

We can also define a supergravity connection

$$\widehat{\nabla}_X = \nabla_X + \frac{1}{24}\left(\widetilde{X}.F - 3F.\widetilde{X}\right)$$

to write the supergravity Killing spinor equation as a parallel spinor equation

$$\widehat{\nabla}_X \epsilon = 0.$$

In $D = 11$, we can search for the solutions of type $M_{11} = M_4 \times M_7$.
So, the metric is in the form of $ds_{11}^2 = ds_4^2 + ds_7^2$.
Co-frame basis, 4-form field F and the spinor parameter ϵ are decomposed as

$$e^a = e^\mu \otimes e^i \qquad , \qquad F = F_4 \otimes F_7 \qquad , \qquad \epsilon = \epsilon_4 \otimes \epsilon_7$$

where $\mu = 0, 1, 2, 3$ and $i = 4, 5, ..., 10$.
To have M_4 as Minkowski space-time, we can choose the ansatz $F = 0$.
Field equatios give M_7 as a Ricci-flat manifold and the supergravity Killing spinor equation reduces to parallel spinor equations $\nabla_X \epsilon_4 = 0 = \nabla_X \epsilon_7$. So, M_7 is a Ricci-flat 7-manifold with a parallel spinor which is a G_2-holonomy manifold.
Namely, we have

$$M_{11} = \underbrace{M_4}_{\text{Minkowski}} \times \underbrace{M_7}_{G_2\text{-holonomy}}$$

If we choose $F = \lambda z_4 \otimes F_7$ where λ is a constant, z_4 is the volume form of M_4 and F_7 is a 4-form on M_7 which satisfies (resulting from the field equations)

$$\begin{aligned} dF_7 &= 0 \\ \delta F_7 &= -c * F_7 \end{aligned} \tag{8.3}$$

with c is a constant. Indeed, these conditions say that F_7 is a special CCKY form.
This choice gives that M_4 and M_7 are Einstein manifolds with M_4 corresponds to AdS_4 space-time, and supergravity Killing spinor equation reduces to Killing spinor equations on M_4 and M_7.
This means that M_7 is an Einstein 7-manifold with a Killing spinor which is a weak G_2-holonomy manifold.
So, we have

$$M_{11} = \underbrace{M_4}_{AdS_4} \times \underbrace{M_7}_{\text{weak } G_2\text{-holonomy}}$$

For $AdS_4 \times S^7$ solution, we need to choose $F_7 = 0$.
(see Duff, Nilsson and Pope, Kaluza-Klein supergravity, *Phys. Rep.* 130 (1986) 1)
Similar procedures can be applied to other dimensions and supergravity theories.

8.2 Supergravity Killing forms

We can analyze the equations satisfied by the spinor bilinears of supergravity Kiliing spinors.
For the six dimensional case, we have the supergravity Killing spinor equation

$$\nabla_X \epsilon = -\frac{1}{4} i_X H.\epsilon.$$

The covariant derivative of the spinor bilinears $(\epsilon\bar{\epsilon})_p$ can be calculated as

$$\begin{aligned}
\nabla_X (\epsilon\bar{\epsilon})_p &= ((\nabla_X \epsilon)\bar{\epsilon})_p + \left(\epsilon(\overline{\nabla_X \epsilon})\right)_p \\
&= -\frac{1}{4}\left(i_X H.\epsilon\bar{\epsilon}\right)_p - \frac{1}{4}\left(\epsilon(\overline{i_X H.\epsilon})\right)_p \\
&= -\frac{1}{4}\left(i_X H.\epsilon\bar{\epsilon}\right)_p + \frac{1}{4}\left(\epsilon\bar{\epsilon}.i_X H\right)_p
\end{aligned}$$

where we have used $\overline{i_X H.\epsilon} = \bar{\epsilon}(i_X H)^{\mathcal{J}} = -\bar{\epsilon}.i_X H$ for $\mathcal{J} = \xi$ or $\xi\eta$ since $i_X H$ is a 2-form.
So, we have

$$\begin{aligned}
\nabla_X (\epsilon\bar{\epsilon})_p &= -\frac{1}{4}\left([i_X H, \epsilon\bar{\epsilon}]_{Cl}\right)_p \\
&= -\frac{1}{4}[i_X H, (\epsilon\bar{\epsilon})_p]_{Cl}
\end{aligned}$$

since the Clifford bracket of any form α with a 2-form $i_X H$, we have the equality $[i_X H, \alpha]_{Cl} = -2 i_{X_a} i_X H \wedge i_{X^a}\alpha$.
The equation satisfied by spinor bilinears reduces to the Killing equation for $p = 1$;

$$\begin{aligned}
\nabla_{X_a} (\epsilon\bar{\epsilon})_1 &= -\frac{1}{4}[i_{X_a} H, (\epsilon\bar{\epsilon})_1]_{Cl} \\
&= \frac{1}{2} i_{X_b} i_{X_a} H \wedge i_{X^b} (\epsilon\bar{\epsilon})_1
\end{aligned}$$

by taking wedge product with e^a, we have

$$\begin{aligned}
d(\epsilon\bar{\epsilon})_1 &= e^a \wedge \nabla_{X_a}(\epsilon\bar{\epsilon})_1 \\
&= \frac{1}{2}e^a \wedge i_{X_b} i_{X_a} H \wedge i_{X^b}(\epsilon\bar{\epsilon})_1 \\
&= -i_{X_b} H \wedge i_{X^b}(\epsilon\bar{\epsilon})_1.
\end{aligned}$$

Comparing the last two equalities, we obtain

$$\nabla_{X_a}(\epsilon\bar{\epsilon})_1 = \frac{1}{2} i_{X_a} d(\epsilon\bar{\epsilon})_1$$

which is the Killing equation. This means that Dirac currents of supergravity Killing spinors are Killing vector fields.
However, for higher-degree Dirac currents, we have the equation

$$\nabla_X \omega = -\frac{1}{4}[i_X H, \omega]_{Cl}$$

which is different from the KY equation. For example,

$$\nabla_X(\epsilon\bar{\epsilon})_3 = \frac{1}{2} i_X d(\epsilon\bar{\epsilon})_3 + \frac{1}{4}[H, i_X(\epsilon\bar{\epsilon})_3]_{Cl}.$$

So, we obtain another type of generalization of Killing vector fields to higher-degree forms different from KY forms. These forms will be called as supergravity Killing forms.
Supergravity Killing forms have a Lie algebra structure under the Clifford bracket. Namely, if ω_1 and ω_2 are supergravity Killing forms, then $[\omega_1, \omega_2]_{Cl}$ is also satisfies the supergravity Killing form equation.
This can be seen as follows

$$\begin{aligned}
\nabla_X[\omega_1, \omega_2]_{Cl} &= [\nabla_X \omega_1, \omega_2]_{Cl} + [\omega_1, \nabla_X \omega_2]_{Cl} \\
&= -\frac{1}{4}[[i_X H, \omega_1]_{Cl}, \omega_2]_{Cl} - \frac{1}{4}[\omega_1, [i_X H, \omega_2]_{Cl}]_{Cl} \\
&= -\frac{1}{4}[i_X H.\omega_1, \omega_2]_{Cl} + \frac{1}{4}[\omega_1.i_X H, \omega_2]_{Cl} \\
&\quad -\frac{1}{4}[\omega_1, i_X H.\omega_2]_{Cl} + \frac{1}{4}[\omega_1, \omega_2.i_X H]_{Cl} \\
&= -\frac{1}{4}\left(i_X H.\omega_1.\omega_2 + \omega_2.\omega_1.i_X H - i_X h.\omega_2.\omega_1 - \omega_1.\omega_2.i_X H\right) \\
&= -\frac{1}{4}[i_X H, [\omega_1, \omega_2]_{Cl}]_{Cl}.
\end{aligned}$$

Since the Clifford bracket is antisymmetric and satisfies the Jacobi identity, supergravity Killing forms have a Lie algebra structure.
Since the supergravity Killing forms of ten dimensional type I and heterotic theories are defined from the same supergravity Killing spinor equation, they also have the similar Lie algebra structure.

In ten-dimensional type IIA supergravity, the supergravity Killing spinor equation is

$$\nabla_X \epsilon = -\frac{1}{4} i_X H.\epsilon - \frac{1}{8} e^{\phi} i_X F_2.\epsilon + \frac{1}{8} e^{\phi} (\widetilde{X} \wedge F_4).\epsilon$$

where ϕ is the dilaton scalar field and F_2 and F_4 are 2-form and 4-form gauge fields.
The supergravity Killing forms satisfy the equation

$$\nabla_X \omega = -\frac{1}{8} \left([2i_X H + e^{\phi} i_X F_2 - e^{\phi} \widetilde{X} \wedge F_4, \omega]_{Cl} \right)$$

where ω is an inhomogeneous form.
Solutions of this equation also have a Lie algebra structure under the Clifford bracket.

In eleven-dimensional supergravity, the supergravity Killing spinor equation is

$$\nabla_X \epsilon = -\frac{1}{24} (\widetilde{X}.F - 3F.\widetilde{X}).\epsilon$$

The equation satisfied by the spinor bilinears can be found as

$$\begin{aligned}
\nabla_X (\epsilon\overline{\epsilon})_p &= ((\nabla_X \epsilon)\overline{\epsilon})_p + \left(\epsilon(\overline{\nabla_X \epsilon})\right)_p \\
&= -\frac{1}{24} \left((\widetilde{X}.F - 3F.\widetilde{X}).\epsilon\overline{\epsilon} \right)_p - \frac{1}{24} \left(\epsilon\overline{(\widetilde{X}.F - 3F.\widetilde{X}).\epsilon} \right)_p \\
&= -\frac{1}{24} \left((\widetilde{X}.F - 3F.\widetilde{X}).\epsilon\overline{\epsilon} \right)_p + \frac{1}{24} \left(\epsilon\overline{\epsilon}.(F.\widetilde{X} - 3\widetilde{X}.F) \right)_p
\end{aligned}$$

where we have used $\overline{(\widetilde{X}.F - 3F.\widetilde{X}).\epsilon} = \overline{\epsilon}.(\widetilde{X}.F - 3F.\widetilde{X})^{\mathcal{J}}$ and by choosing $\mathcal{J} = \xi\eta$ and using $(\widetilde{X}.F)^{\xi\eta} = F^{\xi\eta}.\widetilde{X}^{\xi\eta} = -F.\widetilde{X}$ and $(F.\widetilde{X})^{\xi\eta} = \widetilde{X}^{\xi\eta}.F^{\xi\eta} = -\widetilde{X}.F$. So, we have

$$\nabla_X (\epsilon\overline{\epsilon})_p = -\frac{1}{24} \left\{ \left((\widetilde{X}.F - 3F.\widetilde{X}).\epsilon\overline{\epsilon} \right)_p - \left(\epsilon\overline{\epsilon}.(F.\widetilde{X} - 3\widetilde{X}.F) \right)_p \right\}$$

by adding and subtracting $\frac{1}{24}\epsilon\overline{\epsilon}.(4F.\widetilde{X} - 4\widetilde{X}.F) - \frac{1}{24}\epsilon\overline{\epsilon}.(4F.\widetilde{X} - 4\widetilde{X}.F)$, we obtain

$$\nabla_X (\epsilon\overline{\epsilon})_p = -\frac{1}{24} \left([(\widetilde{X}.F - 3F.\widetilde{X}), \epsilon\overline{\epsilon}]_{Cl} \right)_p - \frac{1}{6} \left(\epsilon\overline{\epsilon}.[F, \widetilde{X}]_{Cl} \right)_p .$$

For $p = 1$, this reduces to the Killing equation and the Dirac currents of supergravity Killing spinors correspond to Killing vector fields.
However, for higher-degree forms it is again different from the KY equation and we find another generalization of Killing vector fields to higher-degree forms.
So, the supergravity Killing form equation in eleven dimensions is

$$\nabla_X \omega = -\frac{1}{24} [(\widetilde{X}.F - 3F.\widetilde{X}), \omega]_{Cl} - \frac{1}{6} \omega.[F, \widetilde{X}]_{Cl}.$$

But, in that case, the solutions does not have a Lie algebra structure automatically. We have

$$\begin{aligned}\nabla_X[\omega_1,\omega_2]_{Cl} &= -\frac{1}{24}[\Phi_X,[\omega_1,\omega_2]_{Cl}]_{Cl}-\frac{1}{6}[\omega_1,\omega_2]_{Cl}.\Psi_X \\ &\quad -\frac{1}{6}(\omega_1.\Psi_X.\omega_2-\omega_2.\Psi_X.\omega_1)\end{aligned}$$

where $\Phi_X=\widetilde{X}.F-3F.\widetilde{X}$ and $\Psi_X=[F,\widetilde{X}]_{Cl}=-2i_XF$.
If the condition

$$\omega_1.\Psi_X.\omega_2=\omega_2.\Psi_X.\omega_1$$

is satisfied, then the Clifford bracket of the solutions is again a solution.
(for more details see Ö. Açik and Ü. Ertem, Hidden symmetries and Lie algebra structures from geometric and supergravity Killing spinors, *Class. Quant. Grav.* 33 (2016) 165002).

Lecture 9

Spin Raising and Lowering Operators

9.1 Spin raising and lowering operators

We consider massless and source-free field equations for different spins

$$\begin{array}{lll} \text{spin} - 0 & \longrightarrow & \Delta f - \frac{n-2}{4(n-1)}\mathcal{R}f = 0 \quad \text{conformal Laplace equation} \\ \text{spin} - \frac{1}{2} & \longrightarrow & \not{D}\psi = 0 \quad \text{massless Dirac equation} \\ \text{spin} - 1 & \longrightarrow & \not{d}F = 0 \quad \text{source-free Maxwell equation} \end{array}$$

By using twistor spinors satsifying $\nabla_X u = \frac{1}{n}\widetilde{X}.\not{D}u$, we can construct spin raising and spin lowering operators that transform solutions of spin-s field equations to solutions of spin-$\left(s+\frac{1}{2}\right)$ field equations and vice versa.

i) Spin raising from spin-0 to spin-$\frac{1}{2}$
For a function f satisfying conformal Laplace equation and a twistor spinor u, we can construct a spinor

$$\psi = df.u + \frac{n-2}{n}f\not{D}u.$$

We have shown in Lecture 7 that ψ is a harmonic spinor, namely it satisfies the massless Dirac equation; $\not{D}\psi = 0$.
So, we transform spin-0 solution f to spin-$\frac{1}{2}$ solution ψ via a twistor spinor u.

ii) Spin lowering from spin-$\frac{1}{2}$ to spin-0
In a reverse procedure, we can construct a function from a massless Dirac spinor ψ and a twistor spinor u by using the spinor inner product $(\,,\,)$ as

$$f = (u, \psi).$$

We will show that f is a solution of the conformal Laplace equation. By applying the LAplace-Beltrami operator $\Delta = \nabla^2 = \nabla_{X_a}\nabla_{X^a} - \nabla_{\nabla_{X_a}X^a}$, we obtain

$$\begin{aligned}
\Delta f &= \nabla_{X_a}\nabla_{X^a}(u,\psi) - \nabla_{\nabla_{X_a}X^a}(u,\psi)\\
&= \nabla_{X_a}\big((\nabla_{X^a}u,\psi) + (u,\nabla_{X^a}\psi)\big) - (\nabla_{\nabla_{X_a}X^a}u,\psi) - (u,\nabla_{\nabla_{X_a}X^a}\psi)\\
&= (\nabla_{X_a}\nabla_{X^a}u,\psi) + 2(\nabla_{X^a}u,\nabla_{X_a}\psi) + (u,\nabla_{X_a}\nabla_{X^a}\psi)\\
&\quad -(\nabla_{\nabla_{X_a}X^a}u,\psi) - (u,\nabla_{\nabla_{X_a}X^a}\psi)\\
&= (\nabla_{X_a}\nabla_{X^a}u - \nabla_{\nabla_{X_a}X^a}u,\psi) + 2(\nabla_{X^a}u,\nabla_{X_a}\psi)\\
&\quad +(u,\nabla_{X_a}\nabla_{X^a}\psi - \nabla_{\nabla_{X_a}X^a}\psi)\\
&= (\nabla^2 u,\psi) + 2(\nabla_{X^a}u,\nabla_{X_a}\psi) + (u,\nabla^2\psi).
\end{aligned}$$

The square of the Dirac operator is written as

$$\not{D}^2\psi = \nabla^2\psi - \frac{1}{4}\mathcal{R}\psi$$

for any spinor ψ. So, we have

$$\begin{aligned}
\Delta(u,\psi) &= (\not{D}^2u + \frac{1}{4}\mathcal{R}u,\psi) + 2(\nabla_{X_a}u,\nabla_{X^a}\psi) + (u,\not{D}^2\psi + \frac{1}{4}\mathcal{R}\psi)\\
&= (\not{D}^2u,\psi) + 2(\nabla_{X_a}u,\nabla_{X^a}\psi) + (u,\not{D}^2\psi) + \frac{1}{2}\mathcal{R}(u,\psi).
\end{aligned}$$

From the twistor equation $\nabla_{X_a}u = \frac{1}{n}e_a.\not{D}u$ and the integrability condition of twistor spinors $\not{D}^2u = -\dfrac{n}{4(n-1)}\mathcal{R}u$, we can write

$$\Delta(u,\psi) = \frac{n-2}{4(n-1)}\mathcal{R}(u,\psi) + \frac{2}{n}(e_a.\not{D}u,\nabla_{X^a}\psi) + (u,\not{D}^2\psi)$$

From the identity $(e_a.\not{D}u,\nabla_{X^a}\psi) = (\not{D}u,(e_a)^{\mathcal{J}}.\nabla_{X^a}\psi) = \pm(\not{D}u,\not{D}\psi)$ and since ψ is a massless Dirac spinor $\not{D}\psi = 0$, we have

$$\Delta(u,\psi) - \frac{n-2}{4(n-1)}\mathcal{R}(u,\psi) = 0$$

which means that $f = (u,\psi)$ is a solution of the conformal Laplace equation.
So, we transform the spin-$\frac{1}{2}$ solution ψ to spin-0 solution f via a twistor spinor u.
We can also construct symmetry operators of massless Dirac spinors by first applying spin lowering procedure from $\frac{1}{2} \to 0$ and then applying spin raising procedure from $0 \to \frac{1}{2}$.
These operators are equivalent to the symmetry operators of massless Dirac spinors constructed in Lecture 5.

iii) Spin raising from spin-$\frac{1}{2}$ to spin-1
In even dimensions $n = 2p$, we can construct a spin-1 quantity from a twistor spinor u and a massless Dirac spinor ψ;

$$F = e^b.u \otimes \overline{\nabla_{X_b}\psi} + \frac{n-2}{n}\not{D}u \otimes \overline{\psi} + \psi \otimes \overline{\not{D}u}.$$

Only the p-form component of F will be important for us.
By applying the Hodge-de Rham operator

$$\begin{aligned}
\not{d}F &= e^a.\nabla_{X_a}F \\
&= e^a.\Big(e^b.\nabla_{X_a}u\otimes\overline{\nabla_{X_b}\psi}+e^b.u\otimes\overline{\nabla_{X_a}\nabla_{X_b}\psi}+\frac{n-2}{n}\nabla_{X_a}\not{D}u\otimes\overline{\psi} \\
&\quad +\frac{n-2}{n}\not{D}u\otimes\overline{\nabla_{X_a}\psi}+\nabla_{X_a}\psi\otimes\overline{\not{D}u}+\psi\otimes\overline{\nabla_{X_a}\not{D}u}\Big) \\
&= e^a.e^b.\nabla_{X_a}u\otimes\overline{\nabla_{X_b}\psi}+e^a.e^b.u\otimes\overline{\nabla_{X_a}\nabla_{X_b}\psi}+\frac{n-2}{n}\not{D}^2u\otimes\overline{\psi} \\
&\quad +\frac{n-2}{n}e^a.\not{D}u\otimes\overline{\nabla_{X_a}\psi}+\not{D}\psi\otimes\overline{\not{D}u}+e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}
\end{aligned}$$

where we have used the normal coordinates and the definiton of the Dirac operator $\not{D}=e^a.\nabla_{X_a}$.
From the Clifford algebra identity $e^a.e^b+e^b.e^a=2g^{ab}$, we can write

$$\begin{aligned}
e^a.e^b.\nabla_{X_a}u\otimes\overline{\nabla_{X_b}\psi} &= (2g^{ab}-e^b.e^a).\nabla_{X_a}u\otimes\overline{\nabla_{X_b}\psi} \\
&= 2\nabla_{X_a}u\otimes\overline{\nabla_{X^a}\psi}-e^a.\not{D}u\otimes\overline{\nabla_{X_a}\psi} \\
&= -\frac{n-2}{n}e^a.\not{D}u\otimes\overline{\nabla_{X_a}\psi}
\end{aligned}$$

since u is a twistor spinor $\nabla_{X_a}u=\frac{1}{n}e_a.\not{D}u$ and by using $\not{D}\psi=0$, we have

$$\begin{aligned}
\not{d}F &= e^a.e^b.u\otimes\overline{\nabla_{X_a}\nabla_{X_b}\psi}+\frac{n-2}{n}\not{D}^2\otimes\overline{\psi}+e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u} \\
&= (e^a\wedge e^b).u\otimes\overline{\nabla_{X_a}\nabla_{X_b}\psi}+u\otimes\overline{\nabla_{X_a}\nabla_{X^a}\psi} \\
&\quad +\frac{n-2}{n}\not{D}^2u\otimes\overline{\psi}+e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}
\end{aligned}$$

where we have used $e^a.e^b=e^a\wedge e^b+\delta^a_b$. By antisymmetrizing the first term and from definition of the curvature operator and the Laplacian, we can write

$$\begin{aligned}
\not{d}F &= \frac{1}{2}(e^a\wedge e^b).u\otimes\overline{R(X_a,X_b)\psi}+u\otimes\overline{\nabla^2\psi} \\
&\quad +\frac{n-2}{n}\not{D}^2u\otimes\overline{\psi}+e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}.
\end{aligned}$$

From the identities $R(X_a,X_b)\psi=\frac{1}{2}R_{ab}.\psi$ and $\not{D}^2\psi=\nabla^2\psi-\dfrac{1}{4}\mathcal{R}\psi$, we obtain

$$\begin{aligned}
\not{d}F &= \frac{1}{4}(e^a\wedge e^b).u\otimes\overline{R_{ab}.\psi}+\frac{1}{4}\mathcal{R}u\otimes\overline{\psi} \\
&\quad +\frac{n-2}{n}\not{D}^2u\otimes\overline{\psi}+e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}
\end{aligned}$$

From the pairwise symmetry of the curvature tensor $R_{abcd}=R_{cdab}$, we can write

$$(e^a\wedge e^b).u\otimes\overline{R_{ab}.\psi}=R_{ab}.u\otimes\overline{(e^a\wedge e^b).\psi}.$$

The integrability conditions of twistor spinors give that $R_{ab}.u = \frac{2}{n}\left(e_b.\nabla_{X_a}\not{D}u - e_a.\nabla_{X_b}\not{D}u\right)$ and $\not{D}^2u = -\dfrac{n}{4(n-1)}\mathcal{R}u$. So, we have

$$\begin{aligned}
\not{d}F &= \frac{1}{2n}\left(e_b.\nabla_{X_a}\not{D}u - e_a.\nabla_{X_b}\not{D}u\right)\otimes\overline{(e^a\wedge e^b).\psi} + \frac{1}{4(n-1)}\mathcal{R}u\otimes\overline{\psi} + e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}\\
&= \frac{1}{2n}e_b.\nabla_{X_a}\not{D}u\otimes\overline{(e^a.e^b - e^b.e^a).\psi} + \frac{1}{4(n-1)}\mathcal{R}u\otimes\overline{\psi} + e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}\\
&= -\frac{1}{n}e_b.\nabla_{X_a}\not{D}u\otimes\overline{e^b.e^a.\psi} + \frac{1}{n}\not{D}^2u\otimes\overline{\psi} + \frac{1}{4(n-1)}\mathcal{R}u\otimes\overline{\psi} + e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}\\
&= -\frac{1}{n}e_b.\nabla_{X_a}\not{D}u\otimes\overline{e^b.e^a.\psi} + e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}
\end{aligned}$$

where we have used $e^a\wedge e^b = e^a.e^b - \delta^{ab}$ and $e^a.e^b = -e^b.e^a + 2g^{ab}$.
In $n = 2p$ dimensions, the identity $(\psi\otimes\overline{\phi})^\xi = (-1)^{\lfloor r/2\rfloor}\phi\otimes\overline{\psi}$ corresponds to $\psi\otimes\overline{\psi} = \phi\otimes\overline{\psi}$ for the p-form component. Then, we can write

$$\begin{aligned}
e_b.\nabla_{X_a}\not{D}u\otimes\overline{e^b.e^a.\psi} &= e_b.\left(\nabla_{X_a}\not{D}u\otimes\overline{e^b.e^a.\psi}\right)\\
&= e_b.\left(e^b.e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}\right)\\
&= e_b.e^b.e^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}\\
&= ne^a.\psi\otimes\overline{\nabla_{X_a}\not{D}u}
\end{aligned}$$

and this gives the result $\not{d}F = 0$.
So, we obtain a middle-form source-free Maxwell solution F from a massless Dirac spinor ψ via a twistor spinor u.

iv) Spin lowering from spin-1 to spin-$\frac{1}{2}$
In a reverse procedure, we can find a massless Dirac spinor from a middle-form source-free Maxwell solution F by using a twistor spinor u.
Let us define

$$\psi = F.u$$

By applying the Dirac operator

$$\begin{aligned}
\not{D}\psi &= e^a.\nabla_{X_a}(F.u)\\
&= e^a.\nabla_{X_a}F.u + e^a.F.\nabla_{X_a}u\\
&= \not{d}F.u + \frac{1}{n}e^a.F.e_a.\not{D}u
\end{aligned}$$

where we have used the definition of Hodge-de Rham operator and the twistor equation.
Since F is a source-free Maxwell field, we have $\not{d}F = 0$ and for a p-form in $2p$-dimensions we have $e^a.F.e_a = (-1)^p(n-2p)F = 0$. Then, we obtain

$$\not{D}\psi = 0.$$

We can also construct symmetry operators for source-free Maxwell fields by applying spin lowering from $1\to\frac{1}{2}$ and spin raising from $\frac{1}{2}\to 1$.

9.2 Rarita-Schwinger fields

We consider spinor-valued 1-forms representing spin-$\frac{3}{2}$ particles.
For a spinor fields ψ_a and the co-frame basis e^a, we define the spinor-valued 1-form as

$$\Psi = \psi_a \otimes e^a.$$

Action of a Clifford form α on Ψ is defined by

$$\alpha.\Psi = \alpha.\psi_a \otimes e^a.$$

Inner product of a spinor-valued 1-form Ψ and a spinor u is defined by

$$(u, \Psi) = (u, \psi_a)e^a.$$

We can also define the Rarita-Schwinger operator acting on spinor-valued 1-forms

$$\not{\mathbb{D}} = e^a.\nabla_{X_a}.$$

From these definitions, the massless Rarita-Schwinger equations of spin-$\frac{3}{2}$ fields in supergravity can be written for $\Psi = \psi_a \otimes e^a$ as follows

$$\not{\mathbb{D}}\Psi = 0 \tag{9.1}$$
$$e^a.\psi_a = 0. \tag{9.2}$$

The first one can be seen as the generalization of the Dirac equation to spin-$\frac{3}{2}$ fields, and second one is the tracelessness condition.
These equations imply a Lorentz-type condition

$$\nabla_{X^a}\psi_a = 0.$$

i) Spin raising from spin-1 to spin-$\frac{3}{2}$
We construct a massless Rarita-Schwinger field from a source-free Maxwell field and a twistor spinor.
For even dimensions $n = 2p$, we propose the following spinor-valued 1-form from a p-form Maxwell field F and a twistor spinor u

$$\Psi = \left(\nabla_{X_a}F.u - \frac{1}{n}F.e_a.\not{D}u\right) \otimes e^a \tag{9.3}$$

where we have $\psi_a = \nabla_{X_a}F.u - \frac{1}{n}F.e_a.\not{D}u$.
To check the tracelessness condition, we calculate $e^a.\psi_a$

$$\begin{aligned} e^a.\psi_a &= e^a.\nabla_{X_a}F.u - \frac{1}{n}e^a.F.e_a.\not{D}u \\ &= \not{d}F.u - (-1)^p\frac{n-2p}{n}F.\not{D}u \\ &= 0 \end{aligned}$$

where we have used $\not{d}F = 0$ and $e^a.F.e_a = (-1)^p(n-2p)F = 0$ since F is a p-form.
By applying Rarita-Schwinger operator to Ψ,

$$\begin{aligned}
\not{\mathbb{D}}\Psi &= e^b.\nabla_{X_b}\left[\left(\nabla_{X_a}F.u - \frac{1}{n}F.e_a.\not{D}u\right)\otimes e^a\right] \\
&= \left(\not{d}\nabla_{X_a}F.u + e^b.\nabla_{X_a}F.\nabla_{X_b}u - \frac{1}{n}\not{d}F.e_a.\not{D}u - \frac{1}{n}e^b.F.e_a.\nabla_{X_b}\not{D}u\right)\otimes e^a \\
&= \left(\not{d}\nabla_{X_a}F.u - \frac{1}{2}e^b.F.e_a.K_b.u\right)\otimes e^a
\end{aligned}$$

where we have used $\not{d}F = 0$, the twistor equation $\nabla_{X_b}u = \frac{1}{n}e_b.\not{D}u$ and the integrability condition $\nabla_{X_b}\not{D}u = \dfrac{n}{2}K_b.u$.
The identity $[\not{d}, \nabla_{X_a}]F = -e^b.R(X_a, X_b)F = -\dfrac{1}{2}e^b.[R_{ab}, F]_{Cl}$ gives

$$\not{\mathbb{D}}\Psi = \left(-\frac{1}{2}e^b.[R_{ab}, F]_{Cl}.u - \frac{1}{2}e^b.F.e_a.K_b.u\right)\otimes e^a.$$

We can write $R_{ab} = C_{ab} + e_b.K_a - e_a.K_b$ in terms of conformal 2-forms and Schouten 1-forms and by using the integrability condition of twistor spinors $C_{ab}.u = 0$, the identity $e^b.R_{ba} = P_a$, we obtain

$$\not{\mathbb{D}}\Psi = \left(\frac{1}{2}P_a.F.u - e^b.F.e_a.K_b.u\right)\otimes e^a.$$

Then, to have a massless Rarita-Schwinger field, the condition

$$P_a.F.u = 2e^b.F.e_a.K_b.u$$

must be satisfied. From the definition $K_a = \frac{1}{n-2}\left(\frac{\mathcal{R}}{2(n-1)}e_a - P_a\right)$ and Clifford multiplying with e^a from the left, this turns into

$$\left(\frac{n(n-1)-2}{(n-1)(n-2)}\right)\mathcal{R}F.u = 0$$

where we have used $e^a.P_a = \mathcal{R}$, $e^a.e_a = n$, $e^a.e^b = -e^b.e^a + 2g^{ab}$ and $e^a.F.e_a = 0$.
So, from a p-form source-free Maxwell field F and a twistor spinor u which satisfies the condition

$$F.u = 0$$

we can construct a massless Rarita-Schwinger field Ψ as in (9.3).

ii) Spin lowering from spin-$\frac{3}{2}$ to spin-1
In four dimensions, we can also construct a source-free Maxwell field from a massless Rarita-Schwinger field via a twistor spinor satisfying a constraint.

Let us define a 1-form A from a mssless Rarita-Schwinger field $\Psi = \psi_a \otimes e^a$ and a twistor spinor u as

$$A = (u, \Psi) = (u, \psi_a)e^a.$$

We consider the 2-form $F = dA$ which can be written as

$$\begin{aligned} F &= d(u, \Psi) \\ &= \left[\frac{1}{n}(e_b.\not{D}u, \psi_a) + (u, \nabla_{X_b}\psi_a)\right] e^b \wedge e^a \end{aligned}$$

by using $d = e^a \wedge \nabla_{X_a}$ and the twistor equation.
Since F is an exact form, we have $dF = 0$. So, the action of Hodge-de Rham operator gives

$$\begin{aligned} \not{d}F &= dF - \delta F \\ &= i_{X^a}\nabla_{X_a}F. \end{aligned}$$

From a direct calculation by using the properties of the inner product and the Rarita-Schwinger equations, we can find

$$\begin{aligned} \not{d}F &= \left[\frac{1}{n}(\nabla_{X_b}\not{D}u, e^c.\psi_a) - \frac{1}{n}(\nabla_{X_b}\not{D}u, e_a.\psi^b)\right. \\ &\quad \left. +(u, \nabla_{X_b}\nabla_{X^b}\psi_a) - (u, \nabla_{X_b}\nabla_{X_a}\psi^b)\right] e^a. \end{aligned}$$

The integrability conition of twistor spinors $\nabla_{X_b}\not{D}u = \frac{n}{2}K_b.u$, the square of the Dirac operator $\not{D}^2 = \nabla^2 - \frac{1}{4}\mathcal{R}$ and the identities $e^a.e_a = n$, $e^a.P_a = \mathcal{R}$, $e^a.K_a = -\frac{\mathcal{R}}{2(n-1)}$ gives

$$\not{d}F = \left[(u, \frac{n-3}{4(n-1)}\mathcal{R}\psi_a - \frac{1}{2}K_b.e_a.\psi^b - \nabla_{X_b}\nabla_{X_a}\psi^b)\right] e^a.$$

This means that, to obtain a source-free Maxwell field, ψ_a of the Rarita-Schwinger field has to satisfy

$$\nabla_{X_b}\nabla_{X_a}\psi^b = -\frac{1}{2}K_b.e_a.\psi^b + \frac{n-3}{4(n-1)}\mathcal{R}\psi_a$$

which can be simplified by using $\nabla_{X_b}\nabla_{X_a}\psi^b = \nabla_{X_a}\nabla_{X_b}\psi^b + R(X_b, X_a)\psi^b$ and $R(X_b, X_a)\psi^b = \frac{1}{2}R_{ba}.\psi^b$ with $\nabla_{X_a}\psi^a = 0$ as

$$(R_{ba} + K_b.e_a).\psi^b = \frac{n-3}{2(n-1)}\mathcal{R}\psi_a.$$

which is automatically satisfied in a flat background.
The symmetry operators can also be constructed by applying spin lowering and spin raising procedures. (for details see Ö. Açik and Ü. Ertem, Spin raising and lowering operators for Rarita-Schwinger fields, arXiv:1712.01594).

Lecture 10

Topological Insulators

Let us consider a d-dimensional periodic crystal lattice.
Periodicity of the lattice results that the unit cell determines all the properties of the whole lattice.
Fourier transform of the unit cell that is the momentum space unit cell corresponds to the Brillouin zone (BZ) which is a d-dimensional torus T^d (from the periodicity of the lattice).

In tight-binding approximation, energy levels of last orbital electrons determine the energy band structure of the crystal.
If there is a gap in energy leves (between valence and conduction bands), it corresponds to an insulator.
If there is no gap in energy leves (between valence and conduction bands), it corresponds to a metal.

The system is determined by the Bloch Hamiltonian $H(k)$ with eigenstates $|u_n(k)\rangle$ and eigenvalues $E_n(k)$ (n corresponds to the number of bands);

$$H(k)|u_n(k)\rangle = E_n(k)|u_n(k)\rangle$$

$H(k)$ is a Hermitian operator on the Hilbert space $\mathcal{H} = \mathbb{C}^{2n}$ and $|u_n(k)\rangle \in \mathbb{C}^{2n}$.
For each $k \in T^d$, we have $|u_n(k)\rangle \in \mathbb{C}^{2n}$.
So, we have the Bloch bundle locally $T^d \times \mathbb{C}^{2n}$ with fiber $\mathbb{C}^{2n}$ and base space T^d.

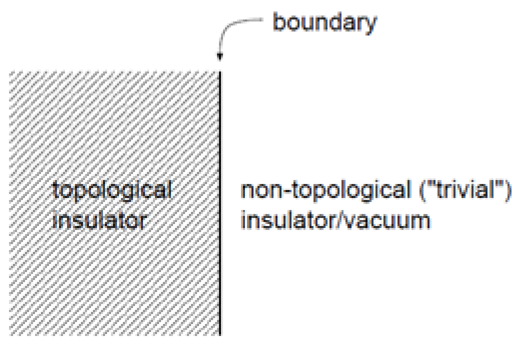
In an insulator, we consider valence Bloch bundle which is locally $T^d \times \mathbb{C}^n$.
All vector bundles are locally trivial; namely can be written as a cross product of base and fiber. However, they can be non-trivial globally.
Triviality or non-triviality of the valence bundle of an insulator determine the topological character of it.

For example; a cylinder $S = S^1 \times \mathbb{R}$ is a trivial line bundle over S^1.
However, a Möbius strip M is a non-trivial line bundle over S^1. It has a twist on the bundle structure.

One can define a section without zero in trivial bundles. But, thisi is not possible in non-trivial bundles.
Non-triviality of a bundle is characterized by the characteristic classes (Chern classes, Pontryagin classes, etc.) which are topological invariants.

A topological insulator is an insulator with a non-trivial valence bundle.
A topological insulator Hamiltonian in one topological class cannot be deformed continuously to a Hamiltonian in another topological class (deformation means changing Hamiltonian parameters without closing the gap).
To convert one topological class Hamiltonian to another one, there must be a gapless state between two classes. Hence, the insulating phase must disappear.

For example, a topological insulator which has an interface with vacuum (which is a trivial insulator) has a gapless boundary although insulating in the bulk.



These gapless boundary degrees of freedom are robust to perturbations.

There are two types of topological insulators;
i) Chern insulators (determined by $\mathbb{Z}$ topological invariants)
ii) $\mathbb{Z}_2$ insulators (determined by $\mathbb{Z}_2$ topological invariants)

10.1 Chern insulators

For the Bloch Hamiltonian $H(k)$ with eigenstates $|u_n(k)\rangle$ and eigenvalues $E_n(k)$;

$$H(k)|u_n(k)\rangle = E_n(k|u_n(k)\rangle)$$

the Berry connection is defined by

$$A = -i\langle u(k)|d|u(k)\rangle$$

where d is the exterior derivative, so A is a 1-form.
The Berry curvature is

$$F = dA.$$

The topological invariant that characterizes the 2-dimensional Chern insulators is the first Chern number

$$C_1 = \frac{1}{2\pi}\int_{BZ} F.$$

For higher even dimensions, we have higher Chern numbers (which take integer values).
For example, a Bloch Hamiltonian in the form

$$H(k) = \mathbf{d}(k).\sigma$$

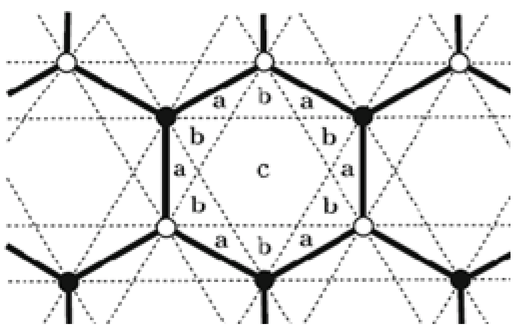
with σ_i are Pauli matrices, the first Chern number corresponds to

$$C_1 = \frac{1}{4\pi}\int_{BZ} d^2k \left(\partial_{k_x}\widehat{\mathbf{d}}(k) \times \partial_{k_y}\widehat{\mathbf{d}}(k)\right).\widehat{\mathbf{d}}(k)$$

where $\widehat{\mathbf{d}}(k) = \frac{\mathbf{d}(\mathbf{k})}{|\mathbf{d}(k)|}$ with $|\mathbf{d}(k)| = \sqrt{d_1^2(k) + d_2^2(k) + d_3^2(k)}$.

Example: Haldane model

Let us consider the honeycomb lattice of graphene with a magnetic flux which is non-zero locally but zero in a unit cell.



The flux ϕ_a in the regions a and the flux ϕ_b in the regions b are related as $\phi_a = -\phi_b$ and the flux $\phi_c = 0$ in the region c. Hence, the total flux is zero.
Bloch Hamiltonian of this system is given by

$$H(k) = \mathbf{d}(k).\sigma$$

with σ_i are Pauli matrices and

$$\begin{aligned}
d_0(k) &= 2t_2 \cos\phi \sum_i \cos(\mathbf{k}.\mathbf{b_i}) \\
d_1(k) &= t_1 \sum_i \cos(\mathbf{k}.\mathbf{a_i}) \\
d_2(k) &= t_1 \sum_i \sin(\mathbf{k}.\mathbf{a_i}) \\
d_3(k) &= M - 2t_2 \sin\phi \sum_i \sin(\mathbf{k}.\mathbf{b_i})
\end{aligned}$$

where $\mathbf{a_i}$ are Bravais lattice basis vectors and $\mathbf{b}_1 = \mathbf{a}_2 - \mathbf{a}_3$, $\mathbf{b}_2 = \mathbf{a}_3 - \mathbf{a}_1$, $\mathbf{b}_3 = \mathbf{a}_1 - \mathbf{a}_2$.
t_1 and t_2 are nearest neighbour and next nearest neighbour hopping parameters, respectively and M is the on-site energy.
$\phi = \frac{2\pi}{\phi_0}(2\phi_a + \phi_b)$ with $\phi_0 = \frac{h}{c}$.
Valence and conduction bands of graphene touch each other at two different points which are called K and K' points.
By considering low energy limit of $H(k)$ (small k limit) at K and K' points, we obtain

$$\begin{aligned}
H(K+k) &= -3t_2\cos\phi + \frac{3}{2}at_1(-k_x\sigma_1 - k_y\sigma_2) + (M + 3\sqrt{3}t_2\sin\phi)\sigma_3 \\
H(K'+k) &= -3t_2\cos\phi + \frac{3}{2}at_1(k_x\sigma_1 - k_y\sigma_2) + (M - 3\sqrt{3}t_2\sin\phi)\sigma_3
\end{aligned}$$

This is a Dirac Hamiltonian and σ_3 is the mass term which determine the gap property of the system.
Chern number of the system corresponds to

$$\begin{aligned}
C_1 &= \frac{1}{2}\big[\mathrm{sgn}(d_3(k) \text{ at } K) - \mathrm{sgn}(d_3(k) \text{ at } K')\big] \\
&= \frac{1}{2}\big[\mathrm{sgn}(M + 3\sqrt{3}t_2\sin\phi) - \mathrm{sgn}(M - 3\sqrt{3}t_2\sin\phi)\big]
\end{aligned}$$

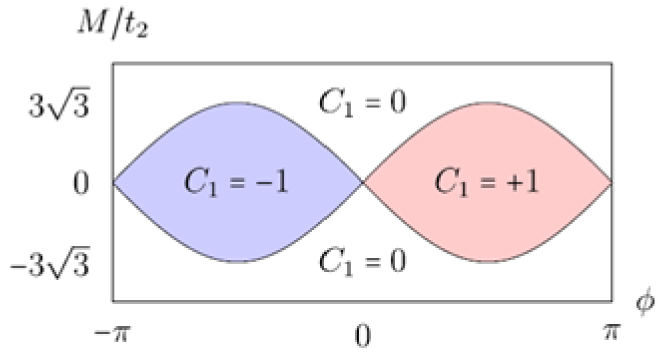
We consider three cases;

i) $M = 3\sqrt{3}t_2\sin\phi$ or $M = -3\sqrt{3}t_2\sin\phi$
In this case $H(k)$ is gapless at K' and gapped at K or gapless at K and gapped at K'.
So, this is a gapless phase.

ii) $M > 3\sqrt{3}t_2\sin\phi$
In this case $C_1 = 0$, hence we have a trivial insulator.

iii) $M < 3\sqrt{3}t_2\sin\phi$
In this case $C_1 = +1$ for $\phi > 0$ and $C_1 = -1$ for $\phi < 0$, hence we have topological insulator phases.

So, by deforming Hamiltonian parameters, we obtain different topological phases and there is a gapless transition between them which is the case (i).



For $\phi = 0$, hence no magnetic fluxes, Haldane model reduces to an ordinary insulator, $C_1 = 0$ in all cases.

10.2 $\mathbb{Z}_2$ insulators

Time reversal operator T maps momentum k to $-k$.
If the Bloch Hamiltonian $H(k)$ of a system is invariant under T, namely

$$H = THT^{-1}$$

then, Chern number cannot characterize the topological property of the system. Some points on BZ are invariant under T. These are called Time Reversal Invariamt Momentum (TRIM) points.

For the eigenstates $|u_n(k)\rangle$, let us define the sewing matrix

$$w_{mn}(k) = \langle u_m(-k)|T|u_n(k)\rangle$$

By using the determinant $\det(w(k))$ and the Pfaffian $\mathrm{Pf}(w(k))$ which is defined as $\mathrm{Pf}^2 = \det$, we can construct the $\mathbb{Z}_2$ invariant of the system as

$$\nu = \prod_{i=\mathrm{TRIM}} \frac{\mathrm{Pf}(w(k_i))}{\sqrt{\det(w(k_i))}}$$

which takes ± 1 values. If $\nu = 1$, we have trivial insulator and if $\nu = -1$, we have topological insulator.

Example: Kane-Mele model

By adding spin-orbit coupling term to the Haldane model, we obtain Kane-Mele model which is a $\mathbb{Z}_2$ insulator.

Spin degrees of freedom splits the Hamiltonian to spin up and spin down parts, and the mass terms of the Bloch Hamiltonian at K and K' points are given by

$$\begin{aligned} h_\uparrow(K) &= (M + 3\sqrt{3}\lambda_{SO})\sigma_3 + ... \\ h_\uparrow(K') &= (M - 3\sqrt{3}\lambda_{SO})\sigma_3 + ... \\ h_\downarrow(K) &= (M - 3\sqrt{3}\lambda_{SO})\sigma_3 + ... \\ h_\downarrow(K') &= (M + 3\sqrt{3}\lambda_{SO})\sigma_3 + ... \end{aligned}$$

where λ_{SO} is the spin-orbit coupling parameter.
Although the total Chern number does not characterize the topology of the system, the difference of spin up and spin down Chern numbers corresponds to the $\mathbb{Z}_2$ invariant.

We have three cases;

i) $M > 3\sqrt{3}\lambda_{SO}$
Chern numbers are

$$\begin{aligned} C_{1\uparrow} &= \frac{1}{2}\big[\mathrm{sgn}(M + 3\sqrt{3}\lambda SO) - \mathrm{sgn}(M - 3\sqrt{3}\lambda SO)\big] = 0 \\ C_{1\downarrow} &= \frac{1}{2}\big[\mathrm{sgn}(M - 3\sqrt{3}\lambda SO) - \mathrm{sgn}(M + 3\sqrt{3}\lambda SO)\big] = 0 \end{aligned}$$

So, we have $C_{1\uparrow} + C_{1\downarrow} = 0$ and $C_{1\uparrow} - C_{1\downarrow} = 0$, and this corresponds to the trivial insulator.

ii) $M < 3\sqrt{3}\lambda_{SO}$
Chern numbers are $C_{1\uparrow} = 1$ and $C_{1\downarrow} = -1$.
So, we have $C_{1\uparrow} + C_{1\downarrow} = 0$, but $C_{1\uparrow} - C_{1\downarrow} \neq 0$.
This corresponds to a topological insulator.

ii) $M = 3\sqrt{3}\lambda_{SO}$
The Hamiltonian is gapless in this case.
Hence, there is gapless phase between two different topological phases.

Spin Chern number (difference of $C_{1\uparrow}$ and $C_{1\downarrow}$) does not work in all systems, so $\mathbb{Z}_2$ invariant is more general than this.

10.3 Classification

Gapped free-fermion Hamiltonians can be classified w.r.t. anti-unitary symmetries they can admit.
We consider two anti-unitary symmetries;
i) Time-reversal $TH(k)T^{-1} = H(-k)$
ii) Charge conjugation (particle-hole) $CH(k)C^{-1} = -H(-k)$
and their combination

iii) Chiral symmetry $SH(k)S^{-1} = -H(k)$

The square of T and C can be ± 1 (for example for fermionic systems we have $T^2 = -1$).
So, we have three possibilities for T; $T = 0$, $T^2 = \pm 1$
and three possibilities for C; $C = 0$, $C^2 = \pm 1$
where 0 means the non-existence of the symmetry.
Moreover, S can be 0 or 1 when both $T = C = 0$.
Then, we have ten possibilities given in the table which corresponds to the Altland-Zirnbauer (AZ) classification of gapped free-fermion Hamiltonians

T	C	S
0	0	0
0	0	1
$+1$	0	0
$+1$	$+1$	1
0	$+1$	0
-1	$+1$	1
-1	0	0
-1	-1	1
0	-1	0
$+1$	-1	1

Indeed, these Hamiltonians are elements of Cartan symmetric spaces and have topological properties in relevant dimensions.
The general picture is given as a peridoic table in the following form

label	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	$+1$	0	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	$+1$	$+1$	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	$+1$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
DIII	-1	$+1$	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
AII	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	-1	-1	1	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	-1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	$+1$	-1	1	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

The table repeats itself after dimension seven.
The first two rows correspond to complex classes and the last eight rows correspond to real classes.
The class A of dimension 2 is the Haldane model, and the class AII of dimension 2 is the Kane-Mele model.

This table originates from the Clifford algebra chessboard.
If we consider the vector spaces $V = \mathbb{R}^{n,s}$ with n negative and s positive generators, the Clifford algebras defined on them correspond to the matrix algebras constructed out of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$.

$Cl_{n,s}$	$n=0$	1	2	3	4	5	6	7
$s=0$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$

where $\mathbb{K}(n)$ denotes $n \times n$ matrices with entries in $\mathbb{K}$. The table repeats itself after dimension seven.
We define

$$Cl_n^* \equiv Cl_{0,n}$$

and because of the isomorphism

$$Cl_{n,s} \cong Cl^*_{s-n(\text{mod } 8)}$$

the Clifford chessboard is also a table for $Cl^*_{s-n(\text{mod } 8)}$.

We consider real vector bundles on a manifold M with a Clifford algebra structure.
Hence, we have Clifford bundles Cl_k^* on M which means that Clifford algebras with k negative generators.
We can define Dirac operators (Hodge-de Rham operators) $\not{D} = e^a.\nabla_{X_a}$ on these bundles.
Index of a Dirac operator is defined by

$$\text{ind}\not{D}_k = \dim(\ker\not{D}_k) - \dim(\text{coker}\not{D}_k).$$

Atiyah-Singer index theorem relates the analytic index of real Dirac operators to the topological invariants of the bundle as

$$\text{ind}(\not{D}_k) \quad = \quad \begin{cases} \dim_{\mathbb{C}}\mathbf{H}_k(\text{mod } 2), & \text{for } k \equiv 1 \text{ (mod 8)} \\ \dim_{\mathbb{H}}\mathbf{H}_k(\text{mod } 2), & \text{for } k \equiv 2 \text{ (mod 8)} \\ \frac{1}{2}\widehat{A}(M), & \text{for } k \equiv 4 \text{ (mod 8)} \\ \widehat{A}(M), & \text{for } k \equiv 0 \text{ (mod 8)} \end{cases}$$

where $\mathbf{H}_k$ is the space of harmonic spinors and $\widehat{A}(M)$ is the $\widehat{A}$-genus of M which is written in terms of Pontryagin classes.
$\widehat{A}$-genus is an integer number and it is an even integer for dim 4(mod 8).
So, the index of $\not{D}_k$ takes values in $\mathbb{Z}_2$ for $k \equiv 1$ and 2(mod 8) and in $\mathbb{Z}$ for $k \equiv 0$ and 4(mod 8).

If we choose $k = s - n$, the Clifford chessboard can be turned into the index table of $\not{D}_{s-n(\text{mod } 8)}$.

Since $\text{ind}(\not{D}_{s-n(\text{mod } 8)})$ corresponds to 0 or takes values in $\mathbb{Z}$ and $\mathbb{Z}_2$, a simple comparison shows that the Clifford chessboard turns into the peroidic table of topological phases of real classes.
These Dirac operators correspond to the Dirac Hamiltonians of topological insulators and the index of Dirac operators determine the topological invariants characterizing topological phases.

Moreover, $\mathbb{Z}$ and $\mathbb{Z}_2$ groups appearing in the index of Dirac oeprators are related to the K-theory groups of vector bundles.
A minimal free-abelian group obtained from a monoid A (group without inverse) is defines as its K-group $K(A)$.
($K(\mathbb{N}) = \mathbb{Z}$ for addition operation and $K(\mathbb{Z}) = \mathbb{Q}$ for multiplication operation).
Isomorphism classes of vector bundles constitute a monoid structure.
If we consider stable equivalence classes, that is for two vector bundles E and F, if we have

$$E \oplus I^n = F \oplus I^m$$

where I^n is a n-dimensional trivial bundle, then we say that E and F are stably equivalent.
Stable equivalence classes constitute a group whichis the K-group of isomorphism classes monoid.
Index of Dirac operators have 1:1 correspondence with K-groups of sphere bundles and these K-groups which are denoted by $K_{\mathbb{R}}^{-(s-n)(\text{mod } 8)}(pt)$ exactly gives the periodic table of real classes.
These bundles correspond to the Bloch bundles of topological insulators (in the continuum limit) and periodic table is a result of K-groups of Bloch bundles.

There are also relations with Groethendieck groups and Clifford algebra extensions which give the symmetry properties w.r.t T, C and S (see Ü. Ertem, Index of Dirac operators and classification of topological insulators, *J. Phys. Commun.* 1 (2017) 035001).

Complex classes are originated from the table of complex Clifford algebras

$\mathbb{C}l_{s-n(\text{mod } 2)}$	$n=0$	1
$s=0$	$\mathbb{C}l_0$	$\mathbb{C}l_1$
1	$\mathbb{C}l_1$	$\mathbb{C}l_0$

and the index theorem

$$\text{ind}(\not{\mathbb{D}}_k) = \begin{cases} \text{Td}(M), & \text{for } k \text{ even} \\ 0, & \text{for } k \text{ odd} \end{cases}$$

where $\text{Td}(M)$ denotes the Todd class which can be written in terms of Chern classes.